A Survey of the Spherical Space Form Problem

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A Survey of the Spherical Space Form Problem

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Preface

The statement of the spherical space form problem in dimension n is this: to classify all manifolds with the sphere $\mathbf{S}^n$ as universal cover. The problem was first stated in this form by H. Hopf in 1925, but it has only been in the past 25 years that real progress has been made in solving it.

This development started with the classification of the possible groups (the periodic groups) which can act freely on complexes homotopic to spheres. (The basic results here are due to Zassenhaus in 1935 and Suzuki in 1955 at the group theory level, and to Cartan-Eilenberg in 1956 at the geometric level.)

The development began to accelerate with the discovery of J. Milnor in 1957 that some of the periodic groups could not act on any sphere. (The first example is the symmetric group $\mathfrak{S}_3$ which is periodic of period 4.) Then R. Swan showed around 1960 that every periodic group acts freely on a finite complex homotopic to $\mathbf{S}^{km-1}$ for some k, where m is the period of the group. In particular, he showed that $\mathfrak{S}_3$ acts freely on a finite complex homotopic to $\mathbf{S}^3$. Swan's results were one of the major beginnings of algebraic K-theory, while Milnor's results led in large measure to the work of C. T. C. Wall on the foundations of non-simply connected surgery theory.

In the past 10 to 12 years these lines of attack have been refined to the point that we now have effective control of the problem—particularly in dimensions greater than 3. This more recent development probably started with the examples of T. Petrie giving the first actions on actual spheres by groups which could not act freely by linear

actions. The simplest example of this kind is the non-abelian group of order 21, the semidirect product of $\mathbf{Z}/7$ by $\mathbf{Z}/3$, which has period 6 and acts freely on $\mathbf{S}^5$. Next, R. Lee [23] extended Milnor's results in a somewhat unexpected way, obtaining both a new proof of Milnor's original result and a (quite extraordinary) extension of it to a new class of groups, the $Q(16a, b, c)$, which are periodic of period 4. He showed that though these groups act freely, indeed linearly, on $\mathbf{S}^7$, they cannot act freely on any sphere $\mathbf{S}^{8k+3}$.

Then C. T. C. Wall, C. B. Thomas-Wall and finally Ib Madsen–Thomas-Wall proved a series of very general results that covered all cases but one last family of groups. These are the $Q(8a, b, c)$ with a, b, and c odd coprime integers. They have period 4, act linearly on $\mathbf{S}^7$, but whether or not they could act on a sphere $\mathbf{S}^{8k+3}$ was now the final question.

Analysis of this last class was initiated in 1978 by Milgram. It rapidly became clear that its structure was far richer than that of the others. In particular all the previous cases had exhibited a type of regularity. If a group did not act in its period dimension minus 1 then all the groups in its family did not act, and conversely. However, for this family examples were constructed which could act on spheres in the dimensions $8k+3$ with $k \geq 1$, and examples which could not. These groups also gave the first examples where Swan's finiteness obstructions were actually non-zero. Indeed the entire attack on these groups demanded new techniques. The basic tool seems to be the structure of units in certain cyclotomic number fields, and the explicit results known so far depend on the study of such units carried out by Milgram in "*Odd index subgroups of units in cyclotomic fields*".

The analysis of the finiteness obstruction was primarily carried out in "*The Swan finiteness obstructions for periodic groups*", while the analysis of the surgery obstruction was

carried out in "*Patching techniques in surgery and the solution of the compact space form problem*", and independently by Ib Madsen [24]. The answers are not completed for all members of the family—this seems to depend on very deep questions in number theory—but they have been reduced to essentially routine calculations whenever these number theoretic questions can be resolved.

Probably the most interesting question remaining open currently is the question of what happens in dimension 3. Once more it is the $Q(8a, b, c)$ which matter. Those which act on spheres $\mathbf{S}^{8k+3}$ for $k>0$ act freely on homology spheres in dimension 3. What kinds of manifolds these homology spheres are would seem to be a basic question in the theory of 3-manifolds, but is, we suspect, very difficult. In particular are any of the groups $Q(8a, b, c)$ with at least two of a, b, and c greater than one, fundamental groups of closed 3-manifolds?

The object of this survey is to detail the development sketched above. The scope of the literature on the problem and related questions makes a general survey of everything impossible without writing a very long paper, so we have had to omit many things of interest and importance. Hopefully the papers quoted in the Reference section will prove helpful for further study. The survey of Wall should also be useful [59]. However, sometimes, as for example in the discussion of the Madsen-Thomas-Wall results, we have found it possible to use more recent results in surgery and homotopy theory to improve their proofs. But by and large we simply quote the basic results from the literature in such a way that the interested reader can (we hope) obtain a useful overview of the problem and the techniques used in its solution.

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Introduction

The topological spherical space form problem is the study of fixed-point free actions of finite groups on spheres. Equivalently, it is the study of space forms, i.e. manifolds whose universal cover is a sphere.

Any fixed-point free map $g: \mathbf{S}^n \to \mathbf{S}^n$ is homotopic to the antipodal map. Hence $\deg(g) = (-1)^{n+1}$. Thus if n is even, the composite of two fixed-point free maps has a fixed point. From this it follows that the only group which acts freely on an even-dimensional sphere is $\mathbf{Z}/2$. We thus restrict our attention to the case where n is odd.

The earliest (and easiest) examples of space forms are the Clifford–Klein manifolds. A Clifford–Klein manifold is a complete Riemannian manifold with constant sectional curvature equal to $+1$. They are of the form $\mathbf{S}^n/G$ where G is a finite group acting freely and orthogonally on $\mathbf{S}^n$. Equivalently, they are given by an orthogonal representation $\rho: G \to O(n+1)$ with $\rho(g)$ having no $+1$ eigenvalue for all non-trivial elements $g \in G$. The classification of Clifford–Klein manifolds is thus a completely algebraic question in group representation theory. A complete solution was given by J. Wolf [65].

Examples The cyclic groups $\mathbf{Z}/n$ act freely and orthogonally on $\mathbf{S}^1$, while the groups

0.1 $$Q(4k) = \{x, y \mid x^{2k} = 1, x^k = y^2, yxy^{-1} = x^{-1}\}$$

are subgroups of the unit quaternions $\mathbf{S}^3$ and so act freely and orthogonally on $\mathbf{S}^3$ via quaternionic multiplication. The groups $Q(4k)$ are often called the binary dihedral groups.

When k is a power of 2, $Q(4k)$ is a generalized quaternion group.

We now describe necessary conditions for a group G to act freely on $\mathbf{S}^{n-1}$.

A free resolution of period n of G is an exact sequence

$$0 \to \mathbf{Z} \xrightarrow{\mu} F_{n-1} \to \cdots \to F_1 \to F_0 \xrightarrow{\varepsilon} \mathbf{Z} \to 0$$

of $\mathbf{Z}G$-modules with the F_i finitely generated and free. Here G acts trivially on the two $\mathbf{Z}$ terms.

PROPOSITION 0.2 *If G acts freely on $\mathbf{S}^{n-1}$ and $n-1$ is odd, then G has a free resolution of period n.*

Proof $\mathbf{S}^{n-1}/G$ is a closed manifold, so it has the homotopy type of a finite CW-complex X with $\dim X = n-1$. (For the case of a topological manifold see Kirby and Siebenmann [20].) The universal cover $\tilde{X}$ has the homotopy type of $\mathbf{S}^{n-1}$. $G = \pi_1(X)$ acts freely on the cells of $\tilde{X}$; this makes the cellular chains $C_i(\tilde{X})$ free $\mathbf{Z}G$-modules. Also note that since $n-1$ is odd, any map $\mathbf{S}^{n-1} \to \mathbf{S}^{n-1}$ without fixed points has degree 1. Thus the action of G on $H_*(\tilde{X}) = H_*(\mathbf{S}^n)$ is trivial. So we have an exact sequence

$$0 \to \mathbf{Z} \to C_{n-1}(\tilde{X}) \to \cdots \to C_1(\tilde{X}) \to C_0(\tilde{X}) \to \mathbf{Z} \to 0$$

which is a free resolution of period n. □

The existence of a free resolution of period n places strong restrictions on the possible π.

COROLLARY 0.3 *If G acts on $\mathbf{S}^{n-1}$ with $n-1$ odd, then*

(a) *The Tate cohomology $\hat{H}^*(G;\mathbf{Z})$ is periodic of period n;*

(b) $\hat{H}^n(G;\mathbf{Z}) = \mathbf{Z}/|G|$;

(c) *All abelian subgroups of G are cyclic.*

Proof By concatenation one can form a complete resolution

$$\cdots \to F_0 \xrightarrow{\mu\varepsilon} F_{n-1} \to \cdots \to F_0 \xrightarrow{\mu\varepsilon} F_{n-1} \to \cdots$$

with an obvious chain automorphism of degree n. This gives (a). (b) follows from (a) and the fact that $\hat{H}^0(G;\mathbf{Z}) = \mathbf{Z}/|G|\ [C-E]$. (c) follows from the Künneth theorem applied to non-cyclic abelian groups. □

Remark 0.4 Cartan and Eilenberg originally proved 0.3 by using the spectral sequence for group actions. They also showed that for any finite group π, (a)$\Leftrightarrow$(b) and that (c) implies $H^m(\pi;\mathbf{Z}) = Z|\pi|$ for some $m > 0$.

So a free action of G on $\mathbf{S}^{n-1}$ implies G has a free resolution of period n, which implies $H^{n+1}(G;\mathbf{Z}) = \mathbf{Z}/|\pi|$, which implies all abelian subgroups of π are cyclic. To determine if an arbitrary group G admits a free action on $\mathbf{S}^{n-1}$ we need an effective procedure which allows us to work backwards through the implications. Such a program has been developed during the last 25 years and will be described in this survey. In fact we discuss techniques to classify the homotopy types of spherical space forms in dimensions ≥ 5. We do not discuss the homeomorphism classes within the homotopy types.

The six sections are

1. The classification of the finite periodic groups,
2. Periodic resolutions,
3. The Swan and Wall obstructions for P-Groups,
4. A brief review of some facts in surgery theory,
5. The surgery problem—general structure,
6. The surgery problems for the groups $Q(8a, b, c)$.

In Section 1 we give a complete classification of finite groups which satisfy the condition that $H^n(G;\mathbf{Z}) = \mathbf{Z}/|G|$. For such groups Swan defined a finiteness obstruction which lies in a quotient of $\tilde{K}_0(\mathbf{Z}G)$. This obstruction vanishes if and only if a free resolution of period n exists. This problem, while still purely algebraic, becomes arithmetic rather than group theoretic. This Swan obstruction

as generalized by Wall is described in Section 2 and Section 3.

Swan showed that the existence of a free resolution of period n of G is equivalent to the existence of a *finite* CW-complex X homotopic to $\mathbf{S}^{n-1}$ on which G acts freely and cellularly. We can then apply the powerful machinery of surgery theory to try to construct a space form from X/G. The surprising and beautiful result is that this too can be reduced to arithmetic questions. This is described in Sections 4, 5, and 6.

SECTION 1

The Classification of the Finite Periodic Groups

In this section we follow Wolf [65] and the reorganization of that discussion by Thomas and Wall [53]. A complete list of periodic groups will be presented organized in a way which makes the discussion of their actions on spheres more systematic.

DEFINITION 1.1 *A finite group G is a P-group if and only if every abelian subgroup is cyclic.*

Every subgroup of a P-group is a P-group. P. A. Smith [44] first proved that a finite group which acts freely on a sphere is a P-group. The next major result was given by H. Cartan and S. Eilenberg (following E. Artin and J. Tate):

THEOREM 1.2 [10] *The following are equivalent:*

(a) *G is a P-group.*

(b) *Every p-Sylow subgroup is either cyclic or generalized quaternion.*

(c) *$H^n(G; \mathbf{Z}) = \mathbf{Z}/|G|$ for some n.*

DEFINITION 1.3 *If G is a P-group, then the period of G is the least n such that $H^n(G; \mathbf{Z}) = \mathbf{Z}/|G|$.*

The period n is always even. (Idea of proof: map to the cohomology of the p-Sylow subgroups and transfer back to G.) Swan gave a geometric characterization of the period of G:

THEOREM 1.4 [47] (a) *The period of G is the least n so that there is a CW-complex X homotopic to $\mathbf{S}^{n-1}$ on which G acts freely and cellularly and so that the induced action in homology is trivial.*

(b) *If G has period n, then G acts freely and cellularly on a finite CW-complex Y homotopic to $\mathbf{S}^{kn-1}$ for some k.*

The number k need not be equal to one, though this was not known until recently. In [31, 33, 12] examples were found for which there was an $X \simeq \mathbf{S}^{n-1}$ on which a P-group acts freely, but X could not be replaced by a finite complex.

P-groups are often called periodic groups because they have periodic cohomology.

THEOREM 1.5 [10] *If G has period n and $\alpha \in H^n(G; \mathbf{Z}) = \mathbf{Z}/|G|$ is an additive generator then*

$$\cup \alpha : H^i(G; \mathbf{Z}) \to H^{i+n}(G; \mathbf{Z})$$

is an isomorphism for $i > 0$.

The cyclic groups $\mathbf{Z}/n$ have period 2 while the generalized quaternion groups have period 4 [10].

PROPOSITION 1.6 *Let G_1, G_2 be P-groups of period n_1 and n_2. Suppose their orders $|G_1|$ and $|G_2|$ are relatively prime; then $G_1 \times G_2$ is a P-group with period the least common multiple of n_1 and n_2.*

Proof Künneth formula for $H^*(G_1 \times G_2; \mathbf{Z})$. □

A slightly more sophisticated way of obtaining new P-groups from simpler ones is given by

PROPOSITION 1.7 *Let G_1, G_2 be as in* 1.6, *and let $\phi : G_1 \to \mathrm{Aut}(G_2)$ be a homomorphism. This defines an action of G_1 on $H^*(G_2; \mathbf{Z})$. If $v \cdot n_2$ is the first multiple of n_2 for which the action on $H^{vn_2}(G_2; \mathbf{Z})$ is trivial, then $G_2 \times_\phi G_1$ is a P-group of period* l.c.m. (vn_2, n_1).

Proof Use the Hochschild–Serre spectral sequence. Since the groups have coprime order it collapses and 1.7 follows directly. □

The complete classification of P-groups was obtained by Zassenhaus [67, 68] in the solvable case, and Suzuki [45]. (See also [65, 53].) We review it now. Our goal is to explicitly present all P-groups, and to give effective means of calculating their periods and p-hyperelementary subgroups.

The earliest classification result was

THEOREM 1.8 (*Burnside*) *If the p-Sylow subgroups of a finite group G are cyclic for all p, then G is metacyclic.*

A metacyclic group is a semidirect product $\mathbf{Z}/a \times_\phi \mathbf{Z}/b$ where a and b are coprime and the twisting is defined by a homomorphism $\phi : \mathbf{Z}/b \to \mathrm{Aut}(\mathbf{Z}/a)$. We use the notation $A(a, b, \phi)$ for $\mathbf{Z}/a \times_\phi \mathbf{Z}/b$. In particular 1.8 and 1.2 imply that a P-group of odd order is metacyclic. On the other hand a metacyclic group $A(a, b, \phi)$ is a P-group whenever a and b are coprime integers, even if a or b is even, and its period is $2 \cdot |\text{image } \phi|$. (This is an easy application of 1.7.) A dihedral group of order $2a$ with a odd is a metacyclic P-group of period 4.

The generalized quaternion groups, using the principle in 1.7, give rise to a family of groups $Q(2^n a, b, c)$, where a, b, c are coprime odd integers and $n \geq 3$. These are semidirect products:

$$0 \to \mathbf{Z}/a \times \mathbf{Z}/b \times \mathbf{Z}/c \to Q(2^n a, b, c) \to Q(2^n) \to 0$$

defined by mapping $Q(2^n)$ into $\mathrm{Aut}(\mathbf{Z}/a \times \mathbf{Z}/b \times \mathbf{Z}/c)$ so that x inverts elements in $\mathbf{Z}/a$ and $\mathbf{Z}/b$ while y inverts elements in $\mathbf{Z}/a$ and $\mathbf{Z}/c$. (Here we are using the presentation of $Q(2^n)$ given in 0.1). The above notation is due to Milnor [28]. He notes that $Q(2^n a, b, c) \simeq Q(2^n a, c, b)$ and that for $n = 3$, $Q(8a, b, c) \simeq Q(8c, a, b)$. The group $Q(2^n a, 1, 1)$ (which is isomorphic to $Q(2^n a)$) embeds in the unit quaternions $\mathbf{S}^3$. Hence $Q(2^n a, 1, 1)$ acts freely on $\mathbf{S}^3$.

It is easy to see that the action of $Q(2^n)$ on $H^4(\mathbf{Z}/abc; \mathbf{Z})$ is trivial, hence by 1.7 each $Q(2^n a, b, c)$ has period 4. However if (i) $n = 3$ and two or more of (a, b, c) are greater than 1, or

(ii) $n > 3$ and b or c is greater than 1, then $Q(2^n a, b, c)$ does not have a free linear action on $\mathbf{S}^{8k+3}$ [65]. But it does have a free linear action on $\mathbf{S}^{8k+7}$, $k \geq 0$. (See 3.8 below.)

By 1.2, a periodic p-group must be cyclic or generalized quaternion. The next level of complexity are the p-hyperelementary groups. Recall that a p-hyperelementary group is a group G given as a split extension

$$0 \to \mathbf{Z}/n \to G \to G_p \to 0$$

where p is a prime not dividing n and G_p is a p-group. A p-hyperelementary group is completely determined by n, G_p, and $\phi : G_p \to \mathrm{Aut}(\mathbf{Z}/n)$, and will be denoted by $A(n, G_p, \phi)$.

1.2 implies

THEOREM 1.9 *The p-hyperelementary P-groups are the metacyclic groups $A(a, p^r, \phi)$ and the groups $Q(2^n a, b, c) \times \mathbf{Z}/d$.*

The general P-groups are studied by means of their p-hyperelementary subgroups. For instance, there is the result of Swan:

THEOREM 1.10 [48] *Let G be a P-group. The period of G is the least common multiple of the periods of the p-hyperelementary subgroups H.*

We shall see later that p-hyperelementary subgroups play a crucial role in algebraic K- and L-theory due to the induction theorems of Swan and Dress.

We now present the detailed classification of P-groups. The following difficult result is the key to classification in the nonsolvable case:

THEOREM 1.11 (Suzuki)

(a) *The perfect ($G = [G, G]$) P-groups are precisely the groups $SL_2(\mathbf{F}_p)$ where p is a prime greater than 3.*

(b) *A nonsolvable P-group is the extension of* $SL_2(\mathbf{F}_p)$ $(p>3)$ *by a solvable P-group.*

$SL_2(\mathbf{F}_3)$ is also a P-group, indeed it is the binary tetrahedral group, but it is neither perfect nor nonsolvable. $SL_2(\mathbf{F}_5)$ is better known as the binary icosehedral group. Both $SL_2(\mathbf{F}_3)$ and $SL_2(\mathbf{F}_5)$ are subgroups of $\mathbf{S}^3$. $\mathbf{S}^3/SL_2(\mathbf{F}_5)$ is Poincaré's famous example of a homology 3-sphere not homeomorphic to $\mathbf{S}^3$. These groups have presentations as follows:

1.12
$$SL_2(\mathbf{F}_3)=\{x, y \mid x^2=y^3=(xy)^3\}$$
$$SL_2(\mathbf{F}_5)=\{x, y \mid x^2=y^3=(xy)^5\}.$$

The remaining $SL_2(\mathbf{F}_p)$ are less familiar. For $p>5$, $SL_2(\mathbf{F}_p)$ does not act freely and linearly on any sphere [65]. The order of $SL_2(\mathbf{F}_p)$ is $(p^2-1)p$. H. Behr and J. Mennicke [5] give presentations

1.13
$$SL_2(\mathbf{F}_p)_{(p\geq 3)}=\{A, B \mid A^p=1, (AB)^3=B^2, \\ B^4=(A^2BA^{(p+1)/2}B)^3=1\}.$$

Here A can be thought of as $\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}$ and B as $\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$. The outer automorphism group of $\mathrm{SL}_2(\mathbf{F}_p)$ is also known and is always $\mathbf{Z}/2$. The generator acts as conjugation with the matrix $\begin{pmatrix}\omega & 0\\ 0 & 1\end{pmatrix}$ where ω is a non-square (mod p).

From this one can show that there is a non-trivial extension of $SL_2(\mathbf{F}_p)$ for $p\geq 3$ which is a P-group. This is the group which we denote $TL_2(\mathbf{F}_p)$, and it is given as a normal extension

1.14
$$0\to SL_2(\mathbf{F}_p)\to TL_2(\mathbf{F}_p)\to \mathbf{Z}/2\to 0$$

with a new generator g and extension and twisting data given by the formulae

$$1.15 \qquad g^2 = \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix}, \; g\alpha g^{-1} = \begin{pmatrix} \omega & 0 \\ 0 & 1 \end{pmatrix} \alpha \begin{pmatrix} \omega^{-1} & 0 \\ 0 & 1 \end{pmatrix},$$

for all $\alpha \in SL_2$. The only familiar group in this list is $TL_2(\mathbf{F}_3) = O_1^*$, the binary octahedral group.

We may also consider extensions by $SL_2(\mathbf{F}_p)$ which are P-groups. The only time these extensions are not products is when $p = 3$, and we get the groups T_v^* given as normal extensions in two ways:

$$1.16 \qquad \begin{aligned} 0 \to \mathbf{Z}/3^{v-1} \to T_v^* \to SL_2(\mathbf{F}_3) \to 0 \\ 0 \to Q(8) \to T_v^* \to \mathbf{Z}/3^v \to 0 \end{aligned}$$

with presentations

$$\{x, y, z \mid x^2 = y^2 = (xy)^2, z^{3^v} = 1, zxz^{-1} = y, zyz^{-1} = xy\}.$$

Again, each T_v^* has an extension of $\mathbf{Z}/2$ giving the groups O_v^*.

$$1.17 \qquad \begin{aligned} 0 \to T_v^* \to O_v^* \to \mathbf{Z}/2 \to 0 \\ 0 \to \mathbf{Z}/3^{v-1} \to O_v^* \to TL_3(\mathbf{F}_3) \to 0 \end{aligned}$$

with presentation

$$\{T_v^*, w \mid w^2 = x^2, xy^{-1}xw = yx, w^{-1}yw = y^{-1}, w^{-1}zw = z^{-1}\}.$$

The groups T_v^* act freely and linearly on the sphere $\mathbf{S}^3$. Indeed, they admit fixed-point free representations into $U_2(\mathbf{C})$, though for $v > 1$ they do not map into the unit quaternions.

The groups O_v^* admit free actions on $\mathbf{S}^7$ through representations in $U_4(\mathbf{C})$, but they do not act linearly and freely on $\mathbf{S}^3$ for $v > 1$. We will show later that O_v^* has period

4. The Sylow 2-subgroup is $Q(16) \simeq \langle w, x, y \rangle$ with wx being an element of order 8. O_v^* contains the 2-hyperelementary group $Q(16, 3^{v-1}, 1) \simeq (\langle w, x, y, z^3 \rangle)$.

Here now is the classification of the P-groups. For the proof that the following list is complete [53]. Let $O(G)$ be the maximal normal subgroup of odd order of a P-group G. $O(G)$ is necessarily metacyclic. The P-groups break up into six types according to whether $G/O(G)$ is: I. a cyclic 2-group; II. $Q(2^n)$; III. $SL_2(\mathbf{F}_3)$; IV. $TL_2(\mathbf{F}_3)$; V. $SL_2(\mathbf{F}_p)$, $p \geq 5$; VI. $TL_2(\mathbf{F}_p)$, $p \geq 5$.

I. The metacyclic groups $A(m, n, \phi)$ with m, n coprime.

II. The split extensions V (which can be described in three ways):

$$0 \to Q(2^n a, b, c) \times \mathbf{Z}/d \to V \to \mathbf{Z}/e \to 0.$$

1.18
$$0 \to \mathbf{Z}/abcd \to V \to Q(2^n) \times \mathbf{Z}/e \to 0$$
$$0 \to A(abcd, e, \phi) \to V \to Q(2^n) \to 0.$$

III. The split extensions W.

1.19
$$0 \to Q(8) \times \mathbf{Z}/m \to W \to \mathbf{Z}/n \times \mathbf{Z}/3^v \to 0$$
$$0 \to A(m, n, \phi) \to W \to T_v^* \to 0$$

where the $\mathbf{Z}/3^v$ acts on $\mathbf{Z}/m$ as well as extending $Q(8)$ to T_v^*, and m, n, 6 are coprime. $O(W) = A(m, 3^{v-1}n, \psi)$.

IV. The extensions U (the first two are split):

$$0 \to \mathbf{Z}/m \to U \to \mathbf{Z}/n \times O_v^* \to 0.$$

1.20
$$0 \to A(m, n, \phi) \to U \to O_v^* \to 0$$
$$0 \to \mathbf{Z}/m \times T_v^* \to U \to \mathbf{Z}/n \times \mathbf{Z}/2 \to 0$$

where the $\mathbf{Z}/2$ acts some way on $\mathbf{Z}/m$ as well as extending T_v^* to O_v^*. Here, $m, n, 6$ are coprime. $O(U) = A(m, 3^{v-1}n, \psi)$.

V. The products

$$A(m, n, \phi) \times SL_2(\mathbf{F}_p)$$

m, n, (p^2-1) coprime, $p \leq 5$.

VI. The extensions Y (the first two are split):

$$0 \to \mathbf{Z}/m \to Y \to \mathbf{Z}/n \times TL_2(\mathbf{F}_p) \to 0.$$

1.21
$$0 \to A(m, n, \phi) \to Y \to TL_2(\mathbf{F}_p) \to 0$$
$$0 \to A(m, n, \phi) \times SL_2(\mathbf{F}_p) \to Y \to \mathbf{Z}/2 \to 0$$

where the $\mathbf{Z}/2$ acts in some fashion on $\mathbf{Z}/m$ and extends $SL_2(\mathbf{F}_p)$ to $TL_2(\mathbf{F}_p)$. Again m, n, $(p^2-1)p$ coprime, $p \geq 5$.

It is convenient to break class II into three parts.

IIK

1.22 $$0 \to Q(2^n a, 1, 1) \times \mathbf{Z}/d \to V \to \mathbf{Z}/e \to 0.$$

IIL

1.23
$$0 \to Q(2^n a, b, c) \times \mathbf{Z}/d \to V \to \mathbf{Z}/e \to 0$$
$$n > 3, \text{ and } b \text{ or } c > 1.$$

IIM

1.24
$$0 \to Q(8a, b, c) \times \mathbf{Z}/d \to V \to \mathbf{Z}/e \to 0$$
$$\text{two of } a, b, c > 1.$$

It is now our goal to present a list of p-hyperelementary subgroups $\{H_G\}$ of each P-group G. This list is exhaustive in the sense that every p-hyperelementary subgroup H of G is isomorphic to a subgroup of the direct product of H_G with a cyclic group of coprime order.

All p-hyperelementary subgroups of a metacyclic group $A(m, n, \phi)$ are contained in $A(m, p^r, \phi)$ where $p^r | n$. For a general periodic group G we will only list the hyperelementary subgroups which are not contained in the metacyclic

group $O(G)$. For type II there is the 2-hyperelementary group $Q(2^n a, b, c)$. For type III there are $A(m, 3^v, \phi)$ and $Q(8)$. In type IV there is the semidirect product $\mathbf{Z}/m \times_T Q(16, 3^v, 1)$ ($\mathbf{Z}/m \times_T G$ denotes a semidirect product defined by a *twisting* of $\mathbf{Z}/m$ by G). This semidirect product is of type IIK only if $v = 1$ and the action of $\mathbf{Z}/2$ on $\mathbf{Z}/m$ is trivial. Otherwise $\mathbf{Z}/m \times_T Q(16, 3^v, 1)$ is of type IIL. We will break class IV into two parts IV(b) and IV(g) (bad and good respectively) according to whether G contains a group of type IIL or not.

One obtains directly for $SL_2(\mathbf{F}_p)$ that up to conjugacy the q-hyperelementary subgroups are contained in one of the three groups:

$$\mathbf{Z}/p \times_\phi \mathbf{Z}/(p-1) = A(p, p-1, \phi)$$

(here if $x \in \mathbf{Z}/(p-1) \xrightarrow{\alpha} \mathrm{Aut}\,(\mathbf{Z}/p)$, then $\phi(x) = \alpha(2x)$).

$$\begin{aligned} \mathbf{Z}(p-1) \times_T \mathbf{Z}/2 &= \{x, y \mid x^{(p-1)/2} = y^2, yxy^{-1} = x^{-1}\} \\ &= Q(2(p-1)). \end{aligned} \tag{1.25}$$

$$\begin{aligned} \mathbf{Z}/(p+1) \times_T \mathbf{Z}/2 &= \{x, y \mid x^{(p+1)/2} = y^2, yxy^{-1} = x^{-1}\} \\ &= Q(2(p+1)). \end{aligned}$$

This takes care of type V.

Similarly, for $TL_2(\mathbf{F}_p)$ we get the groups:

$$\mathbf{Z}/p \times_T \mathbf{Z}/2(p-1) = A(p, 2(p-1), \phi)$$

(here im $\phi = \mathrm{Aut}\,\mathbf{Z}/p$).

$$\begin{aligned} \mathbf{Z}/2(p-1) \times_T \mathbf{Z}/2 &= \{x, y \mid x^{p-1} = y^2, yxy^{-1} = x^{-1}\} \\ &= Q(4(p-1)) \end{aligned} \tag{1.26}$$

$$\begin{aligned} \mathbf{Z}/2(p \times 1) \times_T \mathbf{Z}/2 &= \{x, y \mid x^{p+1} = y^2, yxy^{-1} = x^{-1}\} \\ &= Q(4(p+1)). \end{aligned}$$

It is convenient to break class VI into two parts VI(g) and VI(b) according to whether the $\mathbf{Z}/2$ acts trivially or non-trivially on $\mathbf{Z}/m$. For type VI(g), 1.26 gives our list of hyperelementary subgroups. For type VI(b), suppose the $\mathbf{Z}/2$ inverts the generator of $\mathbf{Z}/r \subset \mathbf{Z}/m$. Then we have the additional 2-hyperelementary subgroups $Q(4(p-1), r, 1)$ and $Q(4(p+1), r, 1)$. Note that one of these groups has type IIL and the other type IIM.

COROLLARY 1.27 *The period of $SL_2(\mathbf{F}_p)$ is $p-1$ for all $p \geq 5$ while the period of $SL_2(\mathbf{F}_3)$ is 4, and the period of $TL_2(\mathbf{F}_p)$ is $2(p-1)$. The period of O_v^* is 4.*

To calculate the period of a P-group we use 1.10. (Exercise: Make a list of all P-groups of period 4, and compare with Milnor [28].)

A P-group G satisfies Milnor's condition if every element of order 2 is central. **We shall see later that Milnor's condition gives a necessary and sufficient condition for the existence of a free G-action on a sphere of *some* dimension** [27]. For now we record a corollary of the classification:

COROLLARY 1.28 *A P-group not of type I satisfies Milnor's condition.*

We now state J. A. Wolf's results on the existence of orthogonal actions. For the complete classification, see [65].

THEOREM 1.29 *A P-group G admits a free linear action on $\mathbf{S}^n$ for some n if and only if*

(1) *Every subgroup of order pq is cyclic. (p, q = primes)*

and

(2) (a) *G is of type* I, II, III, *or* IV

or (b) *G is of type* V *or* VI *with $p = 5$.*

SECTION 2

Periodic Resolutions

The first question which must be studied in analyzing the space form problem is which P-groups of period n actually act freely on a finite complex $X^{n-1} \simeq \mathbf{S}^{n-1}$. (See the discussion in the introduction and Section 1 for some preliminary remarks.)

The key observation here is that if X^{n-1} is a finite $(n-1)$-dimensional CW-complex with a free cellular G-action, then the cellular chain complex gives an exact sequence

2.1 $$0 \to \mathbf{Z} \to C_{n-1} \to C_{n-2} \to \cdots \to C_1 \to C_0 \to \mathbf{Z} \to 0$$

where each C_j is a finitely generated free $\mathbf{Z}G$-module.

Such sequences may be pasted together to give a periodic resolution of $\mathbf{Z}$ by free modules. Moreover, the map $C_0 = C_n \to \mathbf{Z}$ can be thought of as an n-cocycle in $\mathrm{Hom}(C, \mathbf{Z}) = -\mathrm{Hom}(C_j, \mathbf{Z})$, and hence represents a class $g \in \mathrm{Ext}^n_{\mathbf{Z}G}(\mathbf{Z}, \mathbf{Z}) = H^n(G, \mathbf{Z})$ which turns out to be a unit. In [57] Wall studied the more general question of characterizing sequences 2.1 where the C_j are merely assumed finitely generated projective. This is based on previous work of Swan [47] and Wall [61], and the key invariant lies in the reduced stable projective class group $\tilde{K}_0(\mathbf{Z}G)$.

In this section we review this work.

A. The Swan Subgroup and Periodic Resolutions

Associated to a ring A are abelian groups $K_0(A)$ and $K_1(A)$. Recall $K_0(A)$ is the Grothendieck construction on the set

of isomorphism classes of finitely generated projective A-modules and $K_1(A) = GL(A)/[GL(A), GL(A)]$. $\tilde{K}_0(A)$ is $K_0(A)$ modulo the subgroup generated by free modules. For further details see [29].

A particularly important subgroup of $\tilde{K}_0(\mathbf{Z}G)$ is the group T_G—the Swan subgroup of all projectives P_α obtained as pull-backs in the diagram

2.2
$$\begin{tikzcd} P_\alpha \arrow[r] \arrow[dd] & \mathbf{Z} \arrow[d] \\ & \mathbf{Z}/|G| \arrow[d, "\cdot\alpha"] \\ IG = \mathbf{Z}G/\langle\sum\rangle \arrow[r] & \mathbf{Z}/|G|. \end{tikzcd}$$

Remark 2.3 P_α is isomorphic to $\ker(\varepsilon : \mathbf{Z}G \to \mathbf{Z}/\alpha)$ where ε is the augmentation map. By reversing the arrow $\downarrow\cdot\alpha$ one can show P_α is isomorphic to the $\mathbf{Z}G$-ideal $\langle\beta, \sum\rangle$ where $\alpha\beta \equiv 1 \pmod{|G|}$ and $\sum$ is the norm element $\sum_{g\in G} g$.

LEMMA 2.4 (Ullom) *Let G be a finite group.*

(i) $T_G = 0$ *if G is cyclic.*

(ii) T_G *is a quotient of* $(\mathbf{Z}/|G|)^{\cdot}/\pm 1$.

(iii) *The exponent of T_G divides the Artin exponent*† $A(G)$.

(iv) *If H is a subgroup of G and* $(\alpha, |G|) = 1$, *the restriction* $r_H[P_\alpha] = [P_\alpha] \in \tilde{K}_0(\mathbf{Z}H)$.

---

† Let $R(G)$ be the Grothendieck ring of finitely generated $\mathbf{Q}(G)$-modules. Then $I_H^G : R(H) \to R(G)$ is the induction map for $H \subset G$; Artin's theorem states that

$$\bigoplus_{H \text{ cyclic},\, H\subset G} I_H^G(R(H)) \subset R(G)$$

is of finite index; $A(G)$ is the exponent of the quotient group.

Remark 2.5 If $G = Q(2^n a, b, c) \times \mathbf{Z}/d$ then $A(G) = 4$ if G is of type IIL or IIM and is 2 if G is of type IIK. If $G = A(m, p^n, \phi)$ with the image of ϕ having order p^s then $A(G) = p^s$.

Example 2.6 $T_{Q(8)} \neq 0$ by [16]. R. Swan (*J. Reine Angew Math.*, **342** (1983), p. 139) shows that $[p_{12}] \neq 0$ in $T_{Q(8)} \times \mathbf{Z}/3$. Thus $T_{Q(8)} \times T_{\mathbf{Z}/3}$.

The final thing we need to recall about $\tilde{K}_0(\mathbf{Z}G)$ is the Swan induction theorem:

THEOREM 2.7 [46] *Let S represent all conjugacy classes of p-hyperelementary subgroups of G. Then*

$$\coprod_{H \in S} r_H : \tilde{K}_0(\mathbf{Z}G) \to \coprod_{H \in S} \tilde{K}_0(\mathbf{Z}H)$$

is an injection.

B. Periodic Resolutions

We generalize the sequence 2.1. An exact sequence C,

2.8 $$0 \to M \to P_{n-1} \to P_{n-2} \to \cdots \to P_1 \to P_0 \to L \to 0,$$

with the P_i finitely generated projective A-modules defines (as in 2.1) an element.

2.9 $$g(C) \in \mathrm{Ext}_A^n(L, M).$$

The following result completely characterizes such classes:

THEOREM 2.10 (K. Roggenkamp-C. T. C. Wall [57]) *$\xi \in \mathrm{Ext}_A^n(L, M)$ is represented by a sequence (2.8) with the P_i finitely generated projective if and only if for any A-module N it is the case that $\cup g : \mathrm{Ext}_A^i(M, N) \to \mathrm{Ext}_A^{n+i}(L, N)$ is an isomorphism for $i > 0$ and an epimorphism for $i = 0$.*

(Lemma 1.1 and Theorem 1.2 [57].) □

Associated to (2.8) there is an Euler class

$$\chi(C) = \sum_{i=0}^{n-1} (-1)^i [P_i] \in K_0(A)$$

where $[P_i]$ is the class of P_i in $K_0(A)$, and we have

THEOREM 2.12 (Wall) *The Euler characteristic $\chi(C)$ depends only on $g \in \mathrm{Ext}_A^n(L, M)$ and not on the particular sequence which represents it.*

(See lemma 1.3 in [57].) □

Applying 2.10 with $A = \mathbf{Z}G$ and $L = M = N = \mathbf{Z}$ we see by 1.5 that for G a P-group of period n, there is a sequence

2.13 $$0 \to \mathbf{Z} \to P_{n-1} \to \cdots \to P_1 \to P_0 \to \mathbf{Z} \to 0$$

with the P_i finitely generated projectives. Conversely, if there exists a sequence of the above type for an arbitrary group G, then $H^n(G; \mathbf{Z}) = \mathbf{Z}|G|$ so that G is a P-group of period n. By adding elementary complexes $F \to F$ to 2.13 we obtain an exact sequence

2.14 $$0 \to \mathbf{Z} \to P_{n-1} \to C_{n-2} \to \cdots \to C_1 \to C_0 \to \mathbf{Z} \to 0$$

representing the same g as in 2.13 but with all the C_i finitely generated free and only P_{n-1} projective. Thus

COROLLARY 2.15 (Swan) *There is an exact sequence 2.1 with all the C_i free and finitely generated for a P-group G of period n if and only if $\chi(g) = 0$ for some generator g of $H^n(G; \mathbf{Z})$.*

LEMMA 2.16 (Swan [47]) *Let g_1 and g_2 in $\mathbf{Z}/|G|$ represent additive generators of $H^n(G; \mathbf{Z}) = \mathbf{Z}/|G|$ for a p-group G, then*

(a) $\chi(g_1) - \chi(g_2) = [P_{g_1^{-1}g_2}] \in T_G$

(b) $\chi(\underbrace{g_1 \cup g_1 \cdots \cup g_1}_{i \text{ times}}) = i\chi(g_1)$.

COROLLARY 2.17 (Swan [47]) *Let G be a P-group, then there is a well defined $\sigma_n(G) \in \tilde{K}_0(\mathbf{Z}G)/T_G$ for each dimension n for which $H^n(G;\mathbf{Z}) = \mathbf{Z}/|G|$, so that*

(1) $\sigma_{kn}(G) = k\sigma_n(G)$,

(2) *$\sigma_n(G)$ vanishes if and only if there is an exact sequence 2.1 with the C_i all free and finitely generated.*

(3) *$\sigma_n(G)$ vanishes if and only if there is a finite CW-complex $X^{n-1} \simeq \mathbf{S}^{n-1}$ on which G acts freely.*

The proof of (3) depends on the Milnor–Swan construction given in [47].

Remark 2.18 $\sigma_n(G)$ is the Swan finiteness obstruction, while the $\chi(g)$ are the Wall finiteness obstructions for G.

We also have the naturality properties for $H \subset G$,

$$r_H(\chi(g)) = \chi(r_H(g))$$

$$r_H(\sigma_n(G)) = \sigma_n(H).$$

Hence, using 2.9, this reduces matters to a careful study of the $\chi(g)$ as H runs over all p-hyperelementary subgroups.

C. Homotopy Types

For a spherical space form X of dimension $n-1$, the homotopy type is determined by the first Eilenberg–MacLane k-invariant $k(X) \in H^n(\pi_1 X;\mathbf{Z})$. $k(X)$ is defined in terms of the cellular chains $C(\tilde{X})$ and gives a link between algebraic and geometric information. We will give a correspondence between additive generators $g \in H^n(G;\mathbf{Z}) = \mathbf{Z}/|G|$ and polarized homotopy types of complexes X such that $\pi_1 X = G$ and $\tilde{X} \simeq \mathbf{S}^{n-1}$. The finiteness obstruction $\chi(g)$ is then identified with the Wall finiteness obstruction $\theta(X)$.

Let π be any group. Let $C = \{C_i, \partial\}_{i \geq 0}$ be a chain complex of projective $\mathbf{Z}\pi$-modules with $H_0(C) = \mathbf{Z}$. Let $D = \{D_i, \partial\}_{i \geq 0}$ be a projective $\mathbf{Z}\pi$-resolution of $\mathbf{Z}$. Let $m = \min\{i > 0 \mid H_i(C) \neq 0\}$. Then there is a chain map

$$\begin{tikzcd}
D_{m+1} \arrow[r] \arrow[d, "\alpha"] & D_m \arrow[r] \arrow[d] & \cdots \arrow[r] & D_0 \arrow[r] \arrow[d] & \mathbf{Z} \arrow[r] \arrow[d, "1"] & 0 \\
\ker(\partial : C_m \to C_{m-1}) \arrow[r] & C_m \arrow[r] & \cdots \arrow[r] & C_0 \arrow[r] & \mathbf{Z} \arrow[r] & 0
\end{tikzcd}$$

This gives a cohomology class $k(C) \in H^{m+1}(\pi; H_m(C))$ represented by $\alpha : D_{m+1} \to H_m(C)$.

Let X be a connected CW-complex and

$$m = \min\{i > 1 \mid \pi_i(X) \neq 0\}.$$

Then $k(X) = k(C) \in H^{m+1}(\pi_1 X; \pi_m X)$.There is a geometric interpretation of $k(X)$. Let

$$k(\pi_1 X, 1)_m \to X$$

be a map inducing the identity on π_1. (Here X_m represents the m-skeleton of X.) Then $k(X)$ is the obstruction to extending the map across the $(m+1)$-skeleton.

Let π be a group and ρ a $\mathbf{Z}\pi$-module.

DEFINITION 2.19 *X is a polarized (π, n, ρ)-complex if X is a CW-complex of dimension n with $\pi_i(X) = 0$ for $1 < i < n$, equipped with isomorphisms $\alpha_X : \pi_1 X \to \pi$ and $\beta_X : \pi_n(X) \to \rho$, where β_X is a $\mathbf{Z}\pi$-module map.*

DEFINITION 2.20 *Two polarized (π, n, ρ)-complexes X and Y have the same polarized homotopy type if there is a homotopy equivalence $f : X \to Y$ such that $\alpha_X = \alpha_Y \circ f_*$ and $\beta_X = \beta_Y \circ f_*$.*

The geometric interpretation of $k(X)$ shows that if two polarized (π, n, ρ) have the same k-invariant then there is no obstruction to construction of a homotopy equivalence.

Thus we have

PROPOSITION 2.21 *Two polarized (π, n, ρ)-complexes have the same polarized homotopy type if and only if they have the same Eilenberg–Maclane k-invariant.*

Let G be a P-group with $H^n(G; \mathbf{Z}) = \mathbf{Z}/|G|$. The k-invariant of a projective resolution of a period n is nothing more than the cohomology class $g \in H^n(G, \mathbf{Z})$ given by

$$\begin{array}{l} P_n = P_0 \to P_{n-1} \to \cdots \to P_0 \to \mathbf{Z} \to 0. \\ \quad\;\; \Big\downarrow g \\ \quad\;\; \mathbf{Z} \end{array}$$

It is an additive generator of $H^n(G; \mathbf{Z})$. (This follows from the identification of $\hat{H}_0(G; \mathbf{Z})$ with $\mathbf{Z}/|G|$ and the construction of a complete resolution with the automorphism of degree n.)

If P is a projective module, then there is a free module such that $P \oplus F$ is free. Indeed, choose Q such that $Q \oplus P$ is free. Then

$$P \oplus (Q \oplus P) \oplus (Q \oplus P) \oplus \cdots = (P \oplus Q) \oplus (P \oplus Q) \oplus \cdots.$$

(This trick is known as the Eilenberg swindle.)

LEMMA 2.22 [47] *Given any additive generator g of $H^n(G; \mathbf{Z}) = \mathbf{Z}/|G|$, there is a polarized $(G, n-1, \mathbf{Z})$-complex X with $k(X) = g$.*

Proof By 1.5, 2.10, and the above remark, there is an exact sequence

$$0 \to \mathbf{Z} \to F_{n-1} \to \cdots \to F_0 \to \mathbf{Z} \to 0$$

with k-invariant g and the F_i-free $\mathbf{Z}G$-modules. Let X_2 be a CW-complex with fundamental group G. Let $C_i = C_i(\tilde{X})$. Then by a generalization of Schanuel's lemma [47],

$$\ker(C_2 \to C_1) \oplus D \simeq \ker(F_2 \to F_1) \oplus E,$$

where $D = F_2 \oplus C_1 \oplus F_0$ and $E = C_2 \oplus F_1 \oplus C_0$. The sequences can be spliced together.

$$0 \to \mathbf{Z} \to F_{n-1} \to \cdots \to F_4 \to F_3 \oplus E$$
$$\to C_2 \oplus D \to C_1 \to C_0 \to \mathbf{Z} \to 0.$$

For each generator of the free $\mathbf{Z}G$-module D add a trivial 2-cell to X_2 to obtain Y_2. For each generator α of $F_3 \oplus E$ add a 3-cell to Y_2 with attaching map $\partial(\alpha) \varepsilon H_2(\tilde{Y}_2) = \pi_2(\tilde{Y}_2) = \pi_2(Y_2)$. Continue inductively to construct X. □

In [60, 61] C. T. C. Wall defined his finiteness obstruction. We now review some of the major results of these papers. Let $C = \{C_i, \partial\}$ be a chain complex of projective $\mathbf{Z}\pi$-modules, with π any group.

DEFINITION 2.23 *C is finitely dominated if C is $\mathbf{Z}\pi$-chain homotopic to a finite projective complex $\{P_i, \partial\}_{k \geq i \geq 0}$.* (I.e. the P_i are finitely generated projective.) *C is homotopy finite if C is $\mathbf{Z}\pi$-chain homotopic to a finite free complex.*

It is not difficult to show

PROPOSITION 2.24 *Suppose C is $\mathbf{Z}\pi$-chain homotopic to a finite projective complex $\{P_i, \partial\}_{k \geq i \geq 0}$. Then*

$$\chi(C) = \textstyle\sum_{i=0}^{\infty} (-1)^i [P_i] \in \tilde{K}_0(\mathbf{Z}G)$$

depends only on the chain homotopy type of C. In particular $\chi(C) = 0$ if and only if C is homotopy finite.

Let X be a connected CW-complex.

DEFINITION 2.25 *X is finitely dominated if there is a finite complex Y and maps $i: X \to Y$ and $r: Y \to X$ such that $r \circ i$ is homotopic to the identity. X is homotopy finite if X is homotopic to a finite CW-complex.*

We then have the beautiful theorem of Wall:

THEOREM 2.26 *X is finitely dominated if and only if the $\mathbf{Z}[\pi_1 X]$-chain complex $C_*(\tilde{X})$ is finitely dominated. X is homotopy finite if and only if $C_*(\tilde{X})$ is homotopy finite.*

Define the Wall finiteness obstruction of a finitely dominated complex X to be

$$\theta(X) = \chi(C_*(\tilde{X})) \in \tilde{K}_0(\mathbf{Z}[\pi_1 X]).$$

COROLLARY 2.27 *Let $H^n(G;\mathbf{Z}) = \mathbf{Z}/|G|$. Let X be a polarized $(G, n-1, \mathbf{Z})$-complex. Then X is finitely dominated, $\theta(X) = \chi(k(X)) \in \tilde{K}_0(\mathbf{Z}_G)$.*

Let X be an $(n-1)$-dimensional CW-complex with $\tilde{X} \simeq \mathbf{S}^{n-1}$ and fundamental group isomorphic to G. Then the k-invariant of X in $H^n(G;\mathbf{Z}) = \mathbf{Z}/|G|$ depends on the polarization, that is, on the choice of isomorphism of $\pi_1 X$ with G and $\pi_{n-1}(X)$ with $\mathbf{Z}$. Two different k-invariants of X differ by an element of $\pm\mathrm{im}(\mathrm{Aut}\, G \to \mathrm{Aut}\, H^n(G;\mathbf{Z}))$. Here $\mathrm{Aut}\, H^n(G;\mathbf{Z}) = (\mathbf{Z}/|G|)^{\cdot}$.

Thus a classification of homotopy types of $(n-1)$-dimensional complexes X such that (a) $\pi_1 X$ is isomorphic to G and (b) $\tilde{X} \simeq \mathbf{S}^{n-1}$ is given by elements of $(\mathbf{Z}/G)^{\cdot}/\langle \pm 1, \mathrm{im}\, \mathrm{Aut}(G)\rangle$.

Example 2.28 $Q(2^n)$ $(n \geq 3)$ acts freely on $\mathbf{S}^3$, so there is a $g \in H^4(Q(2^n);\mathbf{Z})$ such that $\chi(g) = 0$. $(\mathbf{Z}/2^n)^{\cdot} = \mathbf{Z}/2 \times \mathbf{Z}/2^{n-2}$ with generators -1 and 3. The image of $\mathrm{Aut}\, G \to \mathrm{Aut}(H^4(Q(2^n);\mathbf{Z}))$ is $\mathbf{Z}/2^{n-3}$, the squares of $(\mathbf{Z}/2^n)^{\cdot}$. Thus there are two homotopy types. So at least one half of the generators of $H^4(Q(2^n);\mathbf{Z})$ have zero finiteness obstruction. By 2.16(a) we see that $T_{Q(2^n)}$ is a quotient of $\mathbf{Z}/2$ with possible generator $[P_3]$. Further analysis ([16] or [57]) shows that in fact $T_{Q(2^n)} = \mathbf{Z}/2$. Geometrically this means there are two homotopy classes of CW-complexes X whose cover is homotopic to $\mathbf{S}^3$ and whose fundamental group is isomorphic to $Q(2^n)$. One homotopy

type has a non-zero finiteness obstruction and hence is not the homotopy type of a manifold.

D. The Swan Obstruction and Reidemeister Torsion

A basic philosophy is to avoid doing computations in K_0 at all costs—do them in K_1 instead! This is accomplished by means of pullback diagrams and Mayer–Vietoris type sequences in K-theory. Another philosophy is the Hasse principle—to solve a problem over $\mathbf{Z}$, solve it first locally at all primes, and then globally. Consider the pullback diagram:

2.29
$$\begin{tikzcd} \mathbf{Z}G \arrow[r] \arrow[d] & \mathbf{Z}\left(\frac{1}{|G|}\right)G \arrow[d] \\ \coprod_{p\,|\,|G|} \hat{\mathbf{Z}}_p G \arrow[r] & \coprod_{p\,|\,|G|} \hat{\mathbf{Q}}_p G. \end{tikzcd}$$

Here $\hat{\mathbf{Z}}_p = \lim \mathbf{Z}/p^n$ is the p-adic completion of $\mathbf{Z}$ and $\hat{\mathbf{Q}}_p$ is its quotient field. For the arithmetic and geometric properties of these rings see [21], [11]. 2.29 is a generalized Karoubi square [49] and hence induces an exact sequence of algebraic K-groups

2.30
$$\begin{aligned} K_1(\mathbf{Z}G) &\to K_1\left(\mathbf{Z}\left(\frac{1}{|G|}\right)G\right) \oplus \underset{p\,|\,|G|}{-} K_1(\hat{\mathbf{Z}}_p G) \\ &\to \coprod_{p\,|\,|G|} K_1(\hat{\mathbf{Q}}_p G) \xrightarrow{\partial} K_0(\mathbf{Z}_G) \\ &\to K_0\left(\mathbf{Z}\left(\frac{1}{|G|}\right)G\right) \oplus \prod_{p\,|\,|G|} K_0(\hat{\mathbf{Z}}_p G) \\ &\to \coprod_{p\,|\,|G|} K_0(\hat{\mathbf{Q}}_p G). \end{aligned}$$

See Milnor [29] for the definition of the boundary map.

Consider now diagram 2.29. Suppose we are given an exact sequence over $\hat{\mathbf{Z}}_p G$

2.31 $$C_p = \{0 \to \hat{\mathbf{Z}}_p \to C_{n+1} \to \cdots \to C_1 \to C_0 \to \hat{\mathbf{Z}}_p \to 0\}$$

for all p dividing $|G|$ with the C_j free. Suppose for fixed j, the ranks over $\hat{\mathbf{Z}}_p G$ are equal for all p. (Actually these sequences are very easy to construct for p-hyperelementary P-groups [31, 57]).

Let $C = -_{p \mid |G|} C_p$. Also construct an exact sequence over $\mathbf{Z}(1/|G|)G$:

2.32

$$D = \left\{0 \to \mathbf{Z}\left(\frac{1}{|G|}\right) \to D_{n-1} \to \cdots \to D_1 \to D_0 \to \mathbf{Z}\left(\frac{1}{|G|}\right) \to 0\right\}.$$

with the D_j free and rank $D_j = \text{rank } C_j$.

Then it is easy to construct chain isomorphisms $\phi_{i,p} : \hat{\mathbf{Q}}_p \otimes D_i \to \hat{\mathbf{Q}}_p \otimes C_i$ so that $\partial \circ \phi_{i,p} = \phi_{i-1,p} \circ \partial$. We form the pullback chain complex.

2.33
$$\begin{tikzcd}
P \arrow[r] \arrow[dd] & D \arrow[d] \\
 & D \otimes \coprod_{p \mid |G|} \hat{\mathbf{Q}}_p G \arrow[d, "\phi"] \\
C \arrow[r] & C \otimes \coprod_{p \mid |G|} \hat{\mathbf{Q}}_p G
\end{tikzcd}$$

(compare diagram 2.29). Then P is a periodic resolution.

Then we see

$$\partial\left(\coprod_{p\|G|}\sum_{i=0}^{n-1}(-1)^i[\phi_{i,p}]\right)=\chi(P) \tag{2.34}$$

and represents $\sigma_n(G)\in\tilde{K}_0(\mathbf{Z}G)/T_G$. See [31, 57] for more details.

Remark 2.35 From the definition of ∂, one can conclude that for a coprime to $|G|$, $\partial(\coprod_{p\|G|}a)=P_a\in T_G$. If $T_p=\{\alpha\in K_1(\hat{\mathbf{Q}}_pG)|\alpha$ is the identity except at the trivial representation $\hat{\mathbf{Q}}_p\}$, it follows that $\partial(\coprod_{p\|G|}T_p)=T_G$.

We now discuss Reidemeister–De Rham torsion in order to clarify the choice of C, D, and ϕ. Let

$$E=\{0\to E_n\to E_{n-1}\to\cdots\to E_1\to E_0\to 0\}$$

be an acyclic based complex of finitely generated free A-modules of ranks $r_n, r_{n-1},\ldots,r_0$. Then, set $k_i=r_i-r_{i-1}+r_{i-2}-\cdots r_0$ and define the canonical based complex of ranks $r_n, r_{n-1},\ldots,r_0$ as

$$C(r_n,\ldots,r_0)$$

$$=\left\{\begin{array}{ccccccc} 0\to A^{k_{n-1}} & \xrightarrow{\mathrm{id}} & A^{k_{n-1}} & A^{k_{n-3}} & & A^{r_1} & \xrightarrow{\mathrm{id}} A^{r_0}\longrightarrow 0 \\ & & \oplus & \oplus & \cdots & \oplus & \\ & & A^{k_{n-2}}\to & A^{k_{n-2}} & & A^{k_1} & \end{array}\right..$$

Then there is an isomorphism of chain complexes $\gamma: E\to C(r_n,\ldots,r_0)$ and since both are based in each dimension, γ_r can be represented by a non-singular element in $GL_{n_r}(A)$. Hence a well defined class in $K_1(A)$ results. Define the Reidemeister torsion of E as

$$\tau(E)=\sum(-1)^r[\gamma_r]\in K_1(A). \tag{2.36}$$

This is independent of two types of operations:

1. Elementary collapses or expansions, (i.e. adding to or removing from C a complex of the form

2.37 $$(0 \to \cdots \to A \xrightarrow{\mathrm{id}} A \to 0 \to \cdots \to 0)).$$

2. Basis change by elementary matrices, (i.e. matrices E where $E - I$ has precisely one non-zero entry).

There is a slight generalization of the above called Reidemeister–De Rham torsion. Suppose $A \to \Lambda$ is a mapping of rings. Now suppose E is a finite free complex over A, not necessarily based or acyclic. Then if $\Lambda \otimes_A E$ is acyclic we define

2.38 $$\tau(A) = \tau(\Lambda \otimes_A E) \in K_1(\Lambda)/K_1(A)$$

where $\Lambda \otimes_A E$ is given a basis induced by an A-basis of E.

Let Λ be the kernel of the augmentation map

$$\varepsilon\colon \coprod_{p \| G|} \hat{\mathbf{Q}}_p G \to \prod_{p \| G|} \hat{\mathbf{Q}}_p.$$

Then

$$\tau(C) \in K_1(\Lambda)/K_1\left(\coprod_{p \| G|} \hat{\mathbf{Z}}_p G\right)$$

and

$$\tau(D) \in K_1(\Lambda)/K_1(\mathbf{Z}(1/|G|)G)$$

are well defined, with the class of $[\phi]$ equal to $\tau(D) - \tau(C)$. One can further show that $\tau(D)$ is trivial and $\tau(C)$ is in fact an invariant of the group G.

We now examine the Mayer–Vietoris exact sequence 2.30 in greater detail. Since $\hat{\mathbf{Z}}_p G$ is a semilocal ring, $K_0(\hat{\mathbf{Z}}_p G)$ is torsion free, as is $K_0(\hat{\mathbf{Q}}_p G)$. Likewise, the image of ∂ in

$K_0(\mathbf{Z}G)$ is torsion, so we have an exact sequence

$$2.39 \qquad \begin{aligned} 0 \to \tilde{D}(G) \to \tilde{K}_0(\mathbf{Z}G)(= \text{Torsion } K_0(\mathbf{Z}G)) \\ \to \text{Torsion } K_0\left(\mathbf{Z}\left(\frac{1}{|G|}\right)G\right) \to 0 \end{aligned}$$

where $\tilde{D}(G) = \text{image } \partial$. Note that $T_G \subset \tilde{D}(G)$.

It is sometimes convenient to break the study of $\tilde{D}(G)$ into two steps. So first define the local defect groups $LD_p(G)$ as the quotients $LD_p(G) = K_1(\hat{\mathbf{Q}}_p G)/\text{im } K_1(\hat{\mathbf{Z}}_p G)$. Then

$$\tilde{D}(G) = \left(\coprod_{p \mid |G|} LD_p(G)\right) \Big/ \text{im } K_1\left(\mathbf{Z}\left(\frac{1}{|G|}\right)G\right).$$

For calculation of the structure of $LD_p(G)$ see [31, 33, 12].

Deeper analysis of the Reidemeister torsion shows

THEOREM 2.40 [31] *If g is a generator of $H^n(G, \mathbf{Z}) = \mathbf{Z}/|G|$, then $\chi(g) \in D(G)$. In other words there is a representative $\sigma \in -_{p \| G|} K_1(\hat{\mathbf{Q}}_p G)$ such that $\partial(\sigma) = \chi(g)$ and all the components of the reduced norm $nr(\sigma)$ are units of the integers of the center of $\Pi_{p \| G|} \hat{\mathbf{Q}}_p G$.*

E. Some Comments on $\mathbf{K}_1$

The previous section was motivated by the philosophy that calculations in K_1 are easier than calculations in K_0. In this section we discuss K_1. Some familiarity with the groups $K_1(A)$ for A a semilocal ring, a semisimple algebra, or maximal order in a semisimple algebra are needed. References for such material are given in [50, 4].

We wish to calculate some of the groups in 2.30. A theorem of Wang [62, 4] implies that the reduced norm map $nr: K_1(\hat{\mathbf{Q}}_p G) \to \text{center } (\hat{\mathbf{Q}}_p G)^{\cdot}$ is an isomorphism.

Recall that a ring A is semilocal if A modulo its Jacobson radical is semisimple Artinian.

$\hat{\mathbf{Z}}_p G$ is semilocal.

LEMMA 2.41 [4] *If A is semilocal, $A^{\cdot} \to K_1(A)$ is a surjection.*

Often $(\hat{\mathbf{Z}}_p G)^{\cdot}$ can be studied by means of a filtration using the Jacobson radical.

The structure of im $K_1(\mathbf{Z}(1/|G|)G)$ is more difficult to describe. We shall do so however since this is the source of the more delicate number theory involved in the calculation of the Swan obstruction for the groups $Q(2^n a, b, c)$.

Let $\mathbf{Q}G = -_i A_i$ be the decomposition of $\mathbf{Q}G$ into a sum of simple algebras A_i.

Let $K_i =$ center A_i and $R_i = K_i \cap \mathbf{Z}(1/|G|)G$.

Note that $\mathbf{Z}(1/|G|)G$ contains all the central idempotents of $\mathbf{Q}G$ so that the center of $\mathbf{Z}(1/|G|)G$ is $-_i R_i$, where R_i is the integral closure of $\mathbf{Z}(1/|G|)$ in K_i.

A summand A_i is symplectic if $\mathbf{R} \otimes_{\mathbf{Q}} A_i$ is a sum of matrix rings over the real quaternions. Note that if A_i is symplectic, the field K_i must be totally real.

DEFINITION 2.42 *If A_i is symplectic, let $U^+(K_i) = \{u \in R_i^{\cdot} \mid v(u) > 0$ for all embeddings $v: K_i \to \mathbf{R}\}$.*

If A_i is not symplectic, let $U^+(K_i) = R_i^{\cdot}$.

The following theorem follows from the Strong Approximation Theorem [49].

The reduced norm map give a surjection

$$nr: GL\left(\mathbf{Z}\left(\frac{1}{|G|}\right)G\right) \to \coprod_i U^+(K_i). \tag{2.43}$$

The calculation of im $(K_1(\mathbf{Z}(1/|G|)G))$ in 2.30 thus reduces to the calculation of units in cyclotomic number fields. The problem of determining totally positive units seems difficult and interesting. For some results, see [33]. Note that by 2.40, it is only necessary to calculate units (or positive units) of the integers of the fields K_i.

SECTION 3

The Swan and Wall Obstructions for P-groups

From 2.18 and 2.7 we see that if the Swan obstruction $\sigma_n(G)$ vanishes then $\sigma_n(H)=0$ for every p-hyperelementary subgroup H. Of course, *a priori* this last condition for $r_n(t)$ may not be sufficient. It does suffice however, to find a $g \in H^n(G; \mathbf{Z})$ so that $r_H(\chi(g))=0$ for all p-hyperelementary subgroups.

Thus we begin by studying the finiteness obstructions for p-hyperelementary P-groups. These are of two types, $A(m, p^t, \phi)$, and $\mathbf{Z}/a \times Q(2^n b, c, d)$.

THEOREM 3.1 (Wall [57]) *Let* $p^s = |\text{image } \phi|$ *for* $G = A(m, p^t, \phi)$. *Then* G *has period* $2p^s$ *and* $\sigma_{2p^s}(G)=0$.

Proof Suppose first $p^s < p^t$, then there is a free linear representation in $O(2p^s)$ defined by inducing up a faithful 2-dimensional representation of $\mathbf{Z}/m \cdot p^{t-s}$. This gives the desired complex on triangulating the orbit space of $\mathbf{S}^{(2p^s-1)}$ under the action.

Now, consider the case $p^s = p^t$. Then there is an evident surjection $S: A(m, p^{s+1}, \phi\pi) \to A(m, p^s, \phi)$ (where $\pi: \mathbf{Z}/p^{s+1} \to \mathbf{Z}/p^s$ is the surjection). Let

$$G' = A(m, p^{s+1}, \phi\pi).$$

Let

3.2 $$C' = \{0 \to \mathbf{Z} \to C_{2p^s-1} \to C_{2p^s-2} \to \cdots \to C_1 \to C_0 \to \mathbf{Z} \to 0\}$$

be a free resolution for G'. If

3.3 $$C = \mathbf{Z}G \otimes_{\mathbf{Z}G'} C',$$

then C is $\mathbf{Z}G$-free but not acyclic. However for primes q other than p, $\hat{\mathbf{Z}}_q \otimes C$ is acyclic. Indeed the surjection $\hat{\mathbf{Z}}_q G' \to \hat{\mathbf{Z}}_q G$ splits, so $\hat{\mathbf{Z}}_q G$ is a projective (hence flat) $\hat{\mathbf{Z}}_q G'$-module. Locally

$$\begin{aligned}\hat{\mathbf{Z}}_p A(m, p^s, \phi) &= \coprod_{v|m} (\hat{\mathbf{Z}}_p \otimes \mathbf{Z}(\zeta_v)) \times_T \mathbf{Z}/p^s \\ &= \hat{\mathbf{Z}}_p[\mathbf{Z}/p^s] \oplus V\end{aligned}$$

and so we have a similar splitting

3.4 $$\hat{\mathbf{Z}}_p \otimes_{\mathbf{Z}} C = (\hat{\mathbf{Z}}_p[\mathbf{Z}/p^s] \otimes C) \oplus (V \otimes C)$$

and one easily sees that $V \otimes C$ is acyclic, but $\hat{\mathbf{Z}}_p(\mathbf{Z}/p^s) \otimes C$ is not. Now, we define a new resolution at p as

3.5 $$R \oplus (V \otimes C)$$

where R is any $\hat{\mathbf{Z}}_p(\mathbf{Z}/p^s)$-free acyclic resolution of $\hat{\mathbf{Z}}_p$ having the same ranks as C in each dimension.

LEMMA 3.5 *There is an R so that the Reidemeister-DeRham torsion over $\hat{\mathbf{Q}}_p$ of any representation $\hat{\mathbf{Q}}_p(\zeta_{p^j})$, $1 \le j \le s$ is the same as that for $\hat{\mathbf{Q}}_p(\zeta_{p^j}) \otimes_{\mathbf{Z}A(m,p^s,\phi)} C$.*

Proof C is a free $\mathbf{Z}[\mathbf{Z}/p^s]$-module, and as such is based equivalent to a complex

$$\begin{aligned}0 \to \mathbf{Z} \to \mathbf{Z}(\mathbf{Z}/p^s) &\xrightarrow{u(T-1)} \mathbf{Z}(\mathbf{Z}/p^s) \xrightarrow{p\sum} \cdots \\ &\xrightarrow{p\sum} \mathbf{Z}(\mathbf{Z}/p^s) \xrightarrow{T-1} \mathbf{Z}(\mathbf{Z}/p^s) \to \mathbf{Z} \to 0\end{aligned}$$

where u is some element in $\mathbf{Z}(\mathbf{Z}/p^s)$ for which the image of u is $\mathrm{I}(\mathbf{Z}/p^s)$ is a unit†. (Here $\sum = 1 + T + T^2 + \cdots +$

† $I(\mathbf{Z}/p^s)$ is the *augmentation ideal*, the kernel of the augmentation $\varepsilon : \mathbf{Z}(\mathbf{Z}/p^s) \to \mathbf{Z}$ defined by $(\sum_i n_i g_i) = \sum_i n_i$.

T^{p^s-1}). Then set R equal to

$$0 \to \hat{\mathbf{Z}}_p \to \hat{\mathbf{Z}}_p(\mathbf{Z}/p^s) \xrightarrow{u(T-1)} \hat{\mathbf{Z}}_p(\mathbf{Z}/p^s) \xrightarrow{\Sigma} \cdots$$
$$\xrightarrow{\Sigma} \hat{\mathbf{Z}}_p(\mathbf{Z}/p^s) \xrightarrow{T-1} \hat{\mathbf{Z}}_p(\mathbf{Z}/p^s) \to \hat{\mathbf{Z}}_p \to 0. \quad \square$$

Now consider the pull-back diagram

3.6
$$\begin{array}{ccc} C_1 & \longrightarrow & \mathbf{Z}\left(\frac{1}{mp}\right) \otimes C \\ \downarrow & & \downarrow \\ \left(\coprod_{q|m} \hat{\mathbf{Z}}_q \otimes C\right) \oplus (R \oplus (V \otimes C)) & \to & \left(\coprod_{q|m} \hat{Q}_q \otimes C\right) \oplus (\hat{\mathbf{Q}}_p \otimes C) \end{array}$$

where the gluing automorphisms are the identity on $\hat{\mathbf{Q}}_q \otimes C$ and $\hat{\mathbf{Q}}_p \otimes V \otimes C$, while at $\hat{\mathbf{Q}}_p \otimes R$ the Reidemeister-DeRham torsions τ of the gluing automorphisms are 1 at $\hat{\mathbf{Q}}_p(\zeta_{p^j})$, $j \geq 1$ and p^{p^s} at the trivial representation $\hat{\mathbf{Q}}_p$. But this torsion τ satisfies $\partial(\tau) = \chi(C_q)$.

On the other hand, τ is $p^s[p \times -_{q|m} 1_q]$ at the trivial representation, and 1 at all the other representations. But $\partial[p \times -_{q|m} 1_q]$ is in T so by Ullom's result 2.4 (iii) and 2.5:

$$\partial p^s \left[p \times \underset{q|m}{-} 1_q \right] = p^s \partial \left[p \times \underset{q|m}{-} 1_q \right] = 0.$$

Thus 3.1 follows. $\square$

In particular $\sigma_4(D_{2r}) = 0$ if r is odd. Thus there is a finite complex X with $\pi_1 X = D_{2r}$ and $\tilde{X} \simeq \mathbf{S}^3$. By Milnor's theorem [28] X cannot have the homotopy type of a manifold. Hence X is an example of a finite Poincaré complex which does not have the homotopy type of a manifold.

We now consider the groups $Q(2^n a, 1, 1)$, the generalized quaternion groups. They all admit free linear action on $\mathbf{S}^3$

obtained via representations into $SU(2) \subset SO(4)$. Specifically use the presentation $\{x, y \mid x^{2a} = y^2 = (xy)^2)\}$, then

$$x \to \begin{pmatrix} \zeta_{4a} & 0 \\ 0 & \zeta_{4a}^{-1} \end{pmatrix}$$

3.7 $$y \to \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

gives the representations, here ζ_v is a primitive vth root of unity. As before this gives, on triangulating the orbit space, a class $\gamma \in H^4(Q(2^n a, 1, 1), \mathbf{Z})$ so that $\chi(\gamma) = 0$.

The groups $Q(2^n a, b, c)$ with at least two of $a, b, c > 1$ are much more complex. These groups all have period 4 and free actions on $\mathbf{S}^7$ using a linear representation into $SU(4)$. Let $\mathbf{Z}/a \times \mathbf{Z}/b \times \mathbf{Z}/c = \langle \mathbf{Z}a \rangle \times \langle \mathbf{Z}b \rangle \times \langle \mathbf{Z}c \rangle$.

$$x \to \begin{pmatrix} 0 & \zeta_{2^{n-2}} & & \\ -\zeta_{2^{n-2}}^{-1} & 0 & & \\ & & 0 & -\zeta_{2^{n-2}}^{-1} \\ & & \zeta_{2^{n-2}} & 0 \end{pmatrix}$$

3.8 $$y \to \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\mathbf{Z}a \to \begin{pmatrix} \zeta_a & & & 0 \\ & \zeta_a & & \\ & & \zeta_a^{-1} & \\ 0 & & & \zeta_a^{-1} \end{pmatrix}$$

$$\mathbf{Z}b \to \begin{pmatrix} \zeta_b^{-1} & & & 0 \\ & \zeta_b^{-1} & & \\ & & \zeta_b & \\ 0 & & & \zeta_b \end{pmatrix}$$

$$\mathbf{Z}c \to \begin{pmatrix} \zeta_c & & & 0 \\ & \zeta_c^{-1} & & \\ & & \zeta_c^{-1} & \\ 0 & & & \zeta_c \end{pmatrix}$$

Thus $2\sigma_4(Q(2^n a, b, c)) = 0$, and we are reduced to looking at the image of $\sigma_4(Q(2^n a, b, c))$ in $\tilde{K}_0(\mathbf{Z}Q(2^n a, b, c))_{(2)}$.

In [31] this image is completely analyzed for $n = 3$. The question is studied for $n > 3$ in [12]. The steps in both papers are similar. First the 2-torsion at the local defect groups is obtained. This turns out to be easy at all $q \mid abc$ [31]. However at 2 there are difficulties if $n > 3$. Partial results for $n > 3$ are given in [12] while complete results for $n = 3$ are given in [31]. Then the torsion τ of a pull-back diagram similar to 3.6 is evaluated in these defect groups. It is never trivial, hence the problem becomes one of finding elements in $K_1(\mathbf{Z}(1/2abc)[Q(2^n a, b, c])$ which cancel τ. Thus the value of $\sigma_4(Q(2^n a, b, c))$ becomes a question about the units in the center of $\mathbf{Z}(1/2abc)[Q(2^n a, b, c)]$. But this center is a direct sum of rings

3.9

$$R(\lambda_1, \lambda_2, \lambda_3) = \mathbf{Z}\left(\frac{1}{2abc}\right)(\lambda_1, \lambda_2, \lambda_3) \subset \mathbf{Z}\left(\frac{1}{2abc}\right)(\zeta_{2^n abc}).$$

In particular each $R(\lambda_1, \lambda_2, \lambda_3)$ is the ring of integers (over $\mathbf{Z}(1/2abc)$) in a cyclotomic number field. Actually, the index of this field in $\mathbf{Q}(\zeta_w)$ for an appropriate w is either 1, 2, or 4.

In general, this question is too hard, even though, since we only need 2-torsion, it would be sufficient to find some odd index subset of these units. But we can make a considerable simplification [33].

THEOREM 3.10 *Let $K_w \subset \mathbf{Q}(\zeta_w)$ be the maximal totally real such that $|k_w : \mathbf{Q}|$ is a power of 2. The behavior of*

$\sigma_4(Q(2^n a, b, c))$ depends only on $U(K_w)$ (group of units in the ring of integers of K_w) for $w \mid 2^n abc$.

Computations are simplified greatly when the cyclotomic units are of odd index in the units or when all totally positive units are squares. For a reference on cyclotomic number theory and the definition of cyclotomic unit, see [63].

The following theorem is an extension of ideas from [33]. Let $E(K)$ denote the units in the ring of integers of a number field K. Let $E^2(K)$ denote the squares in $E(K)$. Let $E^+(K)$ denote the elements of $E(K)$ which are positive at all real places.

THEOREM 3.11 *Let $K/\mathbf{Q}$ be totally real and Galois of degree 2^r.*

(a) *$E^+(K) = E^2(K)$ if and only if there is a $u \in E(K)$ such that, $Nu = -1$. (Here $N = N_{K/\mathbf{Q}}$.)*

(b) *Let H be a subgroup of $E(K)$ of finite index. If there is a $u \in H$ with $Nu = -1$, then $|E(K)/H|$ is odd.*

Let $\lambda_n = \zeta_n + \zeta_n^{-1}$.

COROLLARY 3.12 *If K is K_p, or K_{pq} with the quadratic symbol $\left(\frac{p}{q}\right) = -1$, or the maximal 2-extension of $\mathbf{Q}$ in $\mathbf{Q}[\lambda_{2^n}, \lambda_p|$ with $p \not\equiv 1 \pmod 8$ then $E^2(K) = E^+(K)$ and the cyclotomic units have odd index in $U(K)$.*

Indeed in the first two cases cyclotomic units of norm -1 are given in [33]. In the last case with $p \equiv 5 \pmod 8$,

$$N_{K/\mathbf{Q}} N_{\mathbf{Q}}[\lambda_{2^n}, \lambda_p]/K(\lambda_p - \lambda_{2^n}) = -1.$$

If $p \equiv 3$ or $7 \pmod 8$ then $N_{K/\mathbf{Q}}(-1 - \lambda_{2^n}) = -1$. This last unit gives Weber's result that $E^+(\mathbf{Q}[\lambda_{2^n}]) = E^2(\mathbf{Q}[\lambda_{2^n}])$ and that $h_+(\mathbf{Q}[\lambda_{2^n}])$ is odd.

The proof of 3.11 will depend on the following lemma:

LEMMA 3.13 *Let G be a p-group, $\mathbf{F}_p$ the finite field with p elements, and $\varepsilon : \mathbf{F}_p G \to \mathbf{F}_p$ the augmentation. If $\varepsilon(u) \neq 0$ then $(\mathbf{F}_p G u) = \mathbf{F}_p G$.*

Proof Equivalently we show $I(G) = \ker(\varepsilon)$ is the unique maximal ideal of $\mathbf{F}_p G$. If J is a maximal ideal then $\mathbf{F}_p G/J$ is irreducible, hence $\mathbf{F}_p G/J = \mathbf{F}_p$ (see Serre [43]). Thus $J = I(G)$. □

Proof of Theorem 3.11 Let $G = \mathrm{Gal}(K/\mathbf{Q})$. Define the signature map

$$S : K^{\cdot} \to \mathbf{F}_2 G$$

$$S(\alpha) = \sum_{\alpha \in G} a_\sigma \sigma$$

where

$$a_\sigma = \begin{cases} 0 & \text{if } \sigma(\alpha) > 0 \\ 1 & \text{if } \sigma(\alpha) < 0. \end{cases}$$

This is a map of G-modules. We have

$$1 \to E^+(K) \to E(K) \xrightarrow{S} \mathbf{F}_2 G$$

hence

$$1 \to E^+(K)/E^2(K) \to E(K)/E^2(K) \xrightarrow{S} \mathbf{F}_2 G.$$

By Dirichlet's unit theorem, $E(K)/E^2(K) = (\mathbf{Z}/2)^{2^r}$. Thus $E^+(K) = E^2(K)$ if and only if S is onto. By lemma 3.13, S is onto if and only if there is a unit u with $Nu = -1$.

Now let H be a subgroup of $E(K)$ of finite index. Let $u \in H$ with $Nu = -1$. Then, by the above discussion, $H^+ = H^2$. Let $v \in E(K)$. Let $v^a \in H$ with $a > 0$ minimal. If a were even, then $v^a \in H^+ = H^2$ and hence $v^{a/2} \in H$. This would contradict the minimality of a. Hence $|E(K)/H|$ is odd. □

This allows us to make direct calculations in many cases:

THEOREM 3.14 [33]

(a) *Let* $p \equiv 3(4)$ *then* $\sigma_4(Q(8p, q, 1)) = 0$ *for* q *prime if and only if*

(i) $q \equiv 1(8)$ *or*

(ii) $q \equiv 5(8)$ *but* $p^v \equiv \pm 1 (\bmod q)$ *for some odd* v.

(b) $\sigma_4(Q(8p, 1, 1)) = 0$ *if* $p \equiv q \equiv 1 \pmod 4$ *but* $\left(\frac{p}{q}\right) = -1$.

In particular, $\sigma_4(Q(24, 5, 1)) \neq 0$. *But* $\sigma_4(Q(24, 13, 1)) = 0$.

When $n > 3$ there are several major differences. First, $Q(2^n, p, 1)$ is not a subgroup of the unit quaternions $\mathbf{S}^3$, while $Q(8, p, 1) \simeq Q(8p)$ is. Second, the Swan obstruction $\sigma_4(Q(2^n, p, 1))$ depends on the units in $\mathbf{Z}[\lambda_{2^{n-1}}, \lambda_p]$. Here it is possible for non-cyclotomic units to influence the Swan obstruction (at least if $p \equiv 1 \pmod 8$). We have

THEOREM 3.15 [12] *Let* p *be a prime not congruent to* 1 *modulo* 8. *Then*

(a) $\sigma_4(Q(2^n, p, 1)) = 0$ *if* $p \equiv -1 \pmod{2^{n-1}}$

(b) $\sigma_4(Q(2^n, p, 1)) \neq 0$ *if* $p \not\equiv 1 \pmod{2^{n-1}}$.

In particular $\sigma_4(Q(16, 3, 1)) \neq 0$. This is a group of order 48 which gives the smallest possible group with a nonzero Swan obstruction. It is a subgroup of O_v^* for $v > 1$. The following is true even in the case $p \equiv 1(8)$, $p \not\equiv 1(2^{n-1})$: $\sigma_4(Q(2^n, p, 1)) \neq 0$ if there is a $u \in \mathbf{Z}[\lambda_{2^{n-1}}, \lambda_p]$ of norm -1. However, we do not know of any such examples when $p \equiv 1 \pmod 8$.

COROLLARY 3.16 $\sigma_4(O_v^*) \neq 0$ *for* $v > 1$.

Remark 3.17 If $\sigma_4(Q(2^n a, b, c)) = 0$ then the exponent of T_G in $\tilde{K}_0(\mathbf{Z}Q(2^n a, b, c))$ is 2 rather than 4 given in 2.6. This we see using the automorphisms of $Q(2^n a, b, c)$ to convert a resolution g with $\chi(g) = 0$ to $k^2 g$ with $\chi(k^2 g) = 0$ once more. See Davis [14] for an application of this idea.

Example 3.18 Let $G = Q(8) \times \mathbf{Z}/3$. Choose $g \in H^4(6; \mathbf{Z})$ so that $x(g) = [p_{17}]$. Then by 2.6, $x(g) \neq 0$. However $r_{Q(8)}x(g) = 0$ and $r_{\mathbf{Z}/3}x(g) = 0$.

This completes our discussion for p-hyperelementary groups. Now we turn to a brief discussion of the finiteness obstructions for all P-groups. The following result is a slight improvement of a result of Wall [57]:

THEOREM 3.19 *Let G be a P-group of period n.*

(a) $\sigma_{2n}(G) = 0$.

(b) $\sigma_n(G) = 0$ *if G is of type* I, IIK, III, IV(g), V, VI(g), *or of type* IV(b) *or* VI(b) *with* $p \equiv q \pmod 4$.

(c) *Suppose G is not type* VI(b) *with* $p \equiv 3 \pmod 4$. *Then* $\sigma_n(G) = 0$ *if and only if* $\sigma_n(H) = 0$ *for all 2-hyperelementary subgroups H.*

Proof First assume G is of type I (metacyclic). Then there is a unique maximal p-hyperelementary subgroup H_p for every prime p. By remarks 2.8 and 2.18 we can adjust any generator $g \in H^n(G; \mathbf{Z})$ at the p-primary part only, to guarantee $r_{H_p}(\chi(g)) = 0$ for all p. Then $\chi(g) = 0$ by Swan's induction theorem.

Next suppose that G is not of type I, $H^m(G; \mathbf{Z}) = \mathbf{Z}/|G|$, and that all 2-hyperelementary subgroups admit linear fixed-point free representations of dimension m. This is the case in (a) with $m = 2n$ and in (b) with $m = n$. All periodic groups have a unique maximal p-hyperelementary subgroup H_p for all odd p. We now once again adjust the generator g at the p-primary part to insure $r_{H_p}(\chi(g)) = 0$ for odd p. For types II, III, IV there is a unique maximal 2-hyperelementary subgroup. We then adjust at 2 and the remaining primes to guarantee $\chi(g) = 0$. For types V and VI there are three maximal 2-hyperelementary subgroups. Cohomology calculations show that we can choose g such that $r_{H_j}(\chi(g)) = 0$ for $j = 1, 2, 3$ [57].

Case (c) follows since for G of type II or IV, there is one maximal 2-hyperelementary subgroup. □

Remark 3.20 It is not clear whether 3.19(c) can be extended to the case VI(b), $p \equiv 3 \pmod 4$. However here the period n is congruent to 4 modulo 8 and G contains sub-groups of type IIL. R. Lee [23] proved that such a group cannot act freely on a closed manifold M such that $H_*(M; \mathbf{Z}/2) = H_*(\mathbf{S}^{n-1}; \mathbf{Z}/2)$. Thus, from the point of view of the space form problem we are interested in $\sigma_{2n}(G)$, which is zero.

SECTION 4

A Brief Review of Some Facts in Surgery Theory

Let B_G be the classifying space for stable spherical fiberings [26, 30]. Then G/O is the fiber in the natural map

4.1 $$B_O \to B_G.$$

Similarly G/PL is the fiber in the natural map

4.2 $$B_{PL} \to B_G$$

where B_{PL} classifies stable piecewise linear S^n bundles. The homotopy types of these spaces are described in [26], and their importance lies in the fact that if $(X^n, \partial X^n)$ is an n-dimensional Poincaré pair with a vector bundle (or PL) reduction of the Spivak normal bundle [26], then the sets of homotopy classes of maps

4.3 $$\begin{gathered}[(X^n, \partial X^n), (G/O, *)] \\ [(X^n, \partial X^n), (G/PL, *)]\end{gathered}$$

are in 1-1 correspondence with the differentiable (PL) cobordism classes of degree 1 normal maps over X^n with a given trvialization on ∂X^n.

In case X^n and ∂X^n are simply connected and we have a degree 1 normal problem

4.4 $$\begin{tikzcd} \nu \arrow[r] \arrow[d] & \xi \arrow[d] \\ (M^n, \partial M^n) \arrow[r, "h"'] & (X^n, \partial X^n) \end{tikzcd}$$

with $h|(\partial M^n) \to \partial X^n$ a homotopy equivalence, it is well known that the obstruction to doing surgery on M^n (away from ∂M^n) to obtain a homotopy equivalence within the normal bordism class in 4.3 is an index if $n \equiv 0 \pmod 4$, an Arf invariant if $n \equiv 2 \pmod 4$, and 0 otherwise, $n \geq 5$.

Thus, we say the surgery obstruction groups in this case are $\mathbf{Z}$ if $n \equiv 0(4)$, $\mathbf{Z}/2$ if $n \equiv 2(4)$ and 0 if n is odd, and we denote them by the table

4.5 $$L_n^h(\mathbf{Z}) = \begin{cases} \mathbf{Z} & n \equiv 0 \pmod 4 \\ \mathbf{Z}/2 & n \equiv 2 \pmod 4 \\ 0 & \text{otherwise} \end{cases}$$

In case X^n, ∂X^n are not simply connected the situation is more complex—basically because it is harder to embed spheres in middle dimensions when $\pi_1(M^{2n}) \neq 0$. But Wall [55] defined groups

4.6 $$L_n^h(\mathbf{Z}\pi_1(M)), \quad L_n^s(\mathbf{Z}\pi_1(M))$$

which are algebraically defined, depend only on the fundamental groups and serve to measure the surgery obstructions in the non-simply connected case. (L_*^h measures things up to h-cobordism, and L_*^s measures up to s-cobordism—distinctions which are not present in the simply connected case.) In particular there is a (set) map

4.7 $$s:[(X^n, \partial X^n), (G/(\ ), *)] \to L_n^{h,s}(\mathbf{Z}\pi_1(X))$$

which has image $0 = s(\alpha)$, if and only if the corresponding surgery problem is normally cobordant to a homotopy equivalence of pairs (respectively s-cobordant to a simple homotopy equivalence of pairs), $n \geq 5$.

If there is an α so that $s(\alpha)=0$, then we can consider the sets

$$\begin{aligned} &\mathscr{H}_{PL}((X^n, \partial X^n)), \mathscr{S}_{PL}((X^n, \partial X^n)) \\ &\mathscr{H}_O((X^n, \partial X^n)), \mathscr{S}_O((X^n, \partial X^n)) \end{aligned} \tag{4.8}$$

of h-cobordism classes of degree 1 normal homotopy equivalences of pairs (fixed on ∂X^n) and s-cobordism classes defined similarly. There is then a long exact sequence of sets

$$\begin{aligned} \cdots &\to [(I \times X^n \cdot \partial(I \times X^n)), (G/(\ ), *)] \\ &\to L^h_{n+1}(\mathbf{Z}(\pi_1 X)) \to \mathscr{H}_{(\ )}((X^n, \partial X^n)) \\ &\to [(X^n, \partial X^n), (G/(\ ), *)] \to L^h_n(\mathbf{Z}\pi_1(X)) \end{aligned} \tag{4.9}$$

(similarly for s) which often allows effective calculation of the functors in 4.8.

These $L^{h,s}_*(\ )$ functors are contravariant with respect to subgroups $H \subset \pi_1(X)$. (Just take the surgery problem obtained by taking H-covers.) This gives rise to a restriction map

$$r_*: L^{h,s}_*(\mathbf{Z}\pi_1(X)) \to L^{h,s}_*(\mathbf{Z}H). \tag{4.10}$$

There is also an induction map defined whenever there is a homomorphism $f: \pi_1(X) \to H$:

$$I_*: L^{h,s}_*(\mathbf{Z}\pi_1(X)) \to L^{h,s}_*(\mathbf{Z}H) \tag{4.11}$$

(change the fundamental group of X to H) and r_H, I_H are connected by the usual types of relations (see e.g. [15]). Using these maps Wall proved the basic result [58].

THEOREM 4.12 *Let M^n be a closed compact oriented differentiable or PL manifold with finite fundamental group π, then if*

$$\begin{tikzcd} \nu \arrow[r] \arrow[d] & \xi \arrow[d] \\ N^n \arrow[r] & M^n \end{tikzcd}$$

is a degree 1 *normal map corresponding to* $\alpha \in [M^n, G/(\ )]$, *we have that*

$$s(\alpha) \in \operatorname{im} I_*: L_n^h(\mathbf{Z}H) \to L_n^h(\mathbf{Z}\pi)$$

where H *is the Sylow* 2*-subgroup of* π. *Moreover,* $s(\alpha) \neq 0$ *if and only if* $r_H(s(\alpha)) \neq 0$ *in* $L_n^h(\mathbf{Z}H)$.

(This was proved by demonstrating under the above assumptions, the existence of a factorization

4.13
$$\begin{tikzcd} {[M^n, G/(\ )]} \arrow[r, "s"] \arrow[dr, "f"'] & L_n^h(\mathbf{Z}\pi) \arrow[d, "s'"] \\ & \Omega_n^{O \text{ or } PL}(G/(\ ) \times B_\pi) \end{tikzcd}$$

where $f([\alpha]) = [\alpha \times \omega]$, and

4.14
$$\omega : M^m \to B_\pi$$

classifies the universal covering of M^n.)

In case $\pi = \pi_1(M^n) = \mathbf{Z}/2$ and $n \equiv 3 \pmod 4$ Wall [55] gave a complete formula for determining the maps:

THEOREM 4.15 $L_3^h(\mathbf{Z}(\mathbf{Z}/2)) = \mathbf{Z}/2$, *and there are (primitive) classes* $k_{4i+2} (i \geq 0)$ *in* $H^{4i+2}(G/(\ ); \mathbf{Z}/2)$ *so that*

$$s([\alpha]) = \langle W(M) \cup \textstyle\sum (\omega^*(e))^i$$
$$\cup\, \delta\alpha^*(k_* + \mathrm{Sq}^2 k_\delta + \mathrm{Sq}^2\mathrm{Sq}^2 k^\delta), [M^n]\rangle \cdot A.$$

where $A \in L_3^h(\mathbf{Z}(\mathbf{Z}/2))$ *is the non-zero class,* $W(M)$ *is the total Stiefel-Whitney class,* $k = \sum_i k_{y_{i+2}}$, *and* $e \in H^1(B_{\mathbf{Z}/2}, \mathbf{Z}/2) = H^1(\mathbf{R}P^\infty, \mathbf{Z}/2)$ *is the non-trivial class.*

Ranicki [41] (based on work of Novikov) simplified Wall's definitions of the L-groups. For example $L_{2n}^h(\mathbf{Z}\pi)$ is the quotient of the monoid (under orthogonal direct sum) of isomorphism classes of finitely generated free $\mathbf{Z}\pi$-modules with $(-1)^n$-symmetric non-singular bilinear

forms with quadratic refinements, modulo hyperbolic forms (not much changed from Wall's definition). But $L^h_{2n+1}(\mathbf{Z}\pi)$ is obtained from the set of isomorphism classes of triples (H, K_1, K_2) where H is a $(-1)^n$-symmetric finitely generated free hyperbolic $\mathbf{Z}\pi$-module and K_1, K_2 are kernels, i.e. K_i is $\mathbf{Z}\pi$-free, the quadratic form vanishes on K_i and $K_i^{\perp} = K_i$ (significantly different from Wall's definition). The relations needed are fairly complex though. See [41, 7] for a discussion.

Using these definitions a concerted attack was mounted on the structure of the $L^h_n(\mathbf{Z}\pi)$ groups for π finite by Bak, Pardon, and Carlsson-Milgram and others [3, 35, 36, 7, 8]. In particular, this work led to the results of Hambleton and Milgram [18] which allow the effective calculation of $L^h_*(\mathbf{Z}\pi)$ for π a finite 2-group.

Among the results of [18] are the complete determination of $L^h_3(\mathbf{Z}Q(2^n))$. $L^h_{\mathrm{odd}}(\mathbf{Z}(\mathbf{Z}/2^i))$ had previously been determined by Bak [2], so all the L-groups for the 2-groups needed for the space form problem are now understood.

The result for $n = 3$ is

THEOREM 4.16 *$L^h_3(\mathbf{Z}(\mathbf{Z}/2^n)) = \mathbf{Z}/2$ and the induction map*

$$I_*: L^h_3(\mathbf{Z}(\mathbf{Z}/2^n)) \to L^h_3(\mathbf{Z}(\mathbf{Z}/2))$$

is an isomorphism.

Bak [1] and later Wall also proved

THEOREM 4.17 *$L^h_{2n+1}(\mathbf{Z}\pi) = 0$ if $|\pi|$ is odd.*

More recently Hambleton [17] and Taylor-Williams studied the question of which surgery obstruction classes can be represented on closed manifolds. In our case their result implies

THEOREM 4.18 *Let $\alpha \in L^h_3(\mathbf{Z}Q(2^n))$, then, if α represents a nonzero surgery class on some closed manifold, there is a surjection*

$$\gamma: Q(2^n) \to \mathbf{Z}/2$$

and $I_*(\alpha) \neq 0$ in $L^h_3(\mathbf{Z}(\mathbf{Z}/2))$.

Here are a few further results on the $L_*^h(\mathbf{Z}\pi)$ for π finite.

Perhaps the most basic result is the theorem of A. Dress [15]:

THEOREM 4.19 *For π a finite group*

$$L_{2n+1}^h(\mathbf{Z}\pi) = \varprojlim_{K \subset \pi} L_{2n+1}^h(\mathbf{Z}K)$$

where K runs over the conjugacy classes in π of 2-hyperelementary subgroups, and the maps are the restrictions.

In particular 4.19 *implies the surgery obstruction is zero for a particular problem if and only if its restriction is zero for each 2-hyperelementary subgroup.*

It also reduces the study of $L_{2n+1}^h(\mathbf{Z}\pi)$ to the same question for 2-hyperelementary groups only.

The main methods in the study of the structure of the $L_*^h(\mathbf{Z}\pi)$ groups for 2-hyperelementary π are some exact sequences. First there are additional $L_*^p(\mathbf{Z}\pi)$ groups—based on projective rather than free $\mathbf{Z}\pi$-modules see References 7, 8, 9, 34, 35, 41. These groups were interpreted geometrically by Pederson and Ranicki [38].

The $L_*^p(\mathbf{Z}\pi)$ groups are calculated using localization sequences (loc. cit.)

4.20

$$\cdots \xrightarrow{\partial} L_n^p(\mathbf{Z}\pi) \to L_n^h(\mathbf{Q}\pi) \to L_n^{p,(\mathrm{tor})}(\mathbf{Z}\pi) \xrightarrow{\partial} L_{n-1}^p(\mathbf{Z}\pi) \to \cdots$$

and the crucial fact

4.21 $$L_n^{p,(\mathrm{tor})}(\mathbf{Z}\pi) = \underset{q \text{ prime}}{-} L_n^{h(\mathrm{tor})}(\hat{\mathbf{Z}}_q\pi).$$

Moreover, there are exact sequences for each q,

4.22

$$\xrightarrow{\partial} L_n^h(\hat{\mathbf{Z}}_q\pi) \to L_n^h(\hat{\mathbf{Q}}_q\pi) \to L_n^{h^1(\mathrm{tor})}(\hat{\mathbf{Z}}_q\pi) \to L_{n-j}^h(\hat{\mathbf{Z}}_q\pi) \to \cdots,$$

identifications

4.23 $$L_n^h(\hat{\mathbf{Z}}_q\pi) = L_n^h(\hat{\mathbf{Z}}_q\pi/\mathrm{rad})$$

where rad = rad $\hat{\mathbf{Z}}_q\pi$ is the Jacobson radical (so $\hat{\mathbf{Z}}_q\pi/\mathrm{rad}$ is a semisimple algebra) and a general "Ranicki–Rothenberg" exact sequence [41]

4.24 $$\begin{aligned} \cdots \to L_{i+1}^p(A) &\to H^{i+1}(\mathbf{Z}/2; \tilde{K}_0(A)) \\ &\to L_i^h(A) \to L_i^p(A) \to \cdots \end{aligned}$$

Suppose A is semi-simple. Then $L_{2n}^p(A)$ is the Witt ring of its center and $L_{2n+1}^p(A) = 0$. $\tilde{K}_0(A) = \mathbf{Z}^{m-1}$ where m is the number of simple algebra summands of A. Thus 4.24 is easily used and working backwards gives control of $L_*^{p,(\mathrm{tor})}(\mathbf{Z}\pi)$, hence, from 4.20, good control of $L_*^p(\mathbf{Z}\pi)$. Finally to get control of $L_*^h(\mathbf{Z}\pi)$ we use 4.24 again. A good example of this last procedure is to be found in the work of Hambleton and Milgram [18] and also in the research of Wright [66].

The above discussion shows that for A semisimple

$$L_{2n+1}^h(A) = \frac{\{[P] \in \tilde{K}_0(A) \mid [P] = [P^*]\}}{\text{image } L_{2n+2}^p(A)}.$$

In particular $L_{2n+1}^h(A)$ is a quotient of a subgroup of $\tilde{K}_0(A)$. Let $A = \hat{\mathrm{Z}}_2\pi/\mathrm{rad} = \mathbf{F}_2\pi/\mathrm{rad}$ for a finite group π. Let $\tilde{X}$ be a regular π-cover of a Poincaré complex X of dimension $2n+1$. A can be regarded as a local coefficient system (i.e. a $\pi_1 X$-module). The surgery semicharacteristic [13] is

$$\chi_{1/2}(\tilde{X}; A) = \sum_{i=0}^{n} (-1)^i [H_i(X; A)] \in L_{2n+1}^h(A).$$

THEOREM 4.25 (Davis [13]) *Let $\alpha: (M^{2n+1}, \upsilon_M) \to (X, \xi)$ be a degree one normal map. Then the image of $s(\alpha)$ in $L_{2n+1}^h(A)$ is $\chi_{1/2}(\tilde{M}; A) - \chi_{1/2}(\tilde{X}; A)$ where $\tilde{M}$ is the induced π-cover of M.*

Thus in the odd-dimensional case the difference of semi-characteristics gives a normal bordism invariant defined without preliminary surgery. If $\tilde{X}$ is the universal cover of X then im $(s(\alpha)) \in L^h_{2n+1}(A)$ has a geometric interpretation. Im $(s(\alpha)) = 0$ if and only if α is normally bordant to a $\mathbf{Z}_{(2)}\pi$-homology equivalence.

PROPOSITION 4.26 [13] *Let $\alpha: (M^{2n+1}, \nu_M) \to (X, \xi)$ be a degree 1 normal map. If X has the homotopy type of a closed manifold, then*

$$\chi_{1/2}(\tilde{M}; A) = \chi_{1/2}(\tilde{X}; A).$$

Proof Let π_2 be a Sylow 2-subgroup of π. By Wall's theorem 4.12, $s(\alpha) \in \operatorname{Im}(L^h_{2n+1}(\mathbf{Z}\pi_2) \to L^h_{2n+1}(\mathbf{Z}\pi))$. Consider the commutative diagram

$$\begin{array}{ccc} L^h_{2n+1}(\mathbf{Z}\pi_2) & \to & L^h_{2n+1}(\mathbf{Z}\pi) \\ \downarrow & & \downarrow \\ L^h_{2n+1}(\mathbf{F}_2\pi_2) & \to & L^h_{2n+1}(\mathbf{F}_2\pi) \end{array}$$

$L^h_{2n+1}(\mathbf{F}_2\pi_2) = L^h_{2n+1}(\mathbf{F}_2) = L^p_{2n+1}(\mathbf{F}_2) = 0$. $L^h_{2n+1}(\mathbf{F}_2\pi) = L^h_{2n+1}(A)$. Thus im $s(\alpha) \in L^h_{2n+1}(A)$ is zero. □

The contrapositive of the theorem can give an obstruction to a Poincaré complex having the homotopy type of a closed manifold. Also it leads to strong restrictions on the homology of a manifold with a free π-action:

THEOREM 4.27 [13] *Suppose a finite group π acts freely on a closed manifold M of odd dimension. Then*

$$\chi_{1/2}(M; \mathbf{F}_2\pi/\mathrm{rad}) = \chi_{1/2}(\pi \times_{\pi_2} M; \mathbf{F}_2\pi/\mathrm{rad}).$$

Proof Suppose $|\pi|$ has prime factors $p_1, \ldots, p_r$ and 2. Let π_p be a p-Sylow subgroup of π. Consider the disjoint union of covering maps

$$4.28 \qquad b(M/\pi_2) \quad \mathop{-}_{i=1}^{i} \; b_i(M/\pi_{p_i}) \to M/\pi.$$

$b_i(M/\pi_{p_i})$ denotes $|b_i|$ disjoint copies of M/π_{p_i} with the orientations reversed if b_i is negative. Choose the integers $b, b_1, \ldots, b_r$ such that the map is degree 1 and the b_i are even. 4.28 is covered by a map of tangent bundles, hence of stable normal bundles. The conclusion of the theorem follows by 4.26 and the fact that $L^h_{2n+1}(\mathbf{F}_2\pi/\mathrm{rad})$ has exponent 2. $\square$

SECTION 5

The Surgery Problems—General Structure

Let π be a P-group and suppose $\sigma_n(\pi)=0$, so there is a finite complex X^{n-1} with fundamental group π and universal cover $\tilde{X}^{n-1}\simeq \mathbf{S}^{n-1}$. Then X^{n-1} is a Poincaré duality space [6] and has a fiber homotopy spherical normal bundle, i.e. a map

5.1 $$f: X \to B_{SG}$$

where B_{SG} is the classifying space for (oriented) fiber homotopy spherical fibrations.

THEOREM 5.2 *f above lifts to a map $h: X \to B_{SO}$.*

Proof Let $\pi_p \subset \pi$ be the p-Sylow subgroup and consider the map $f_p: X_{\pi_p} \to X \xrightarrow{f} B_{SG}$. Each f_p classifies the Spivak normal bundle to X_{π_p}. For p odd $\pi_p = \mathbf{Z}/p^j$ and $X_{\pi_p} \simeq L_{p^j}^{n-1}$ for some Lens space. Hence we get lifting here. For $p=2$, $\pi_p = \mathbf{Z}/2^j$ for which we use the previous argument or $\pi_p = Q(2^n \cdot 1, 1, 1)$. But here $(\mathbf{Z}/2^n)^{\cdot}/\text{squares} = \mathbf{Z}/2 \times \mathbf{Z}/2$ with generators $-1, 5$ and since -1 doesn't matter, just 5 is important. On the other hand $T(Q(2^n 1, 1)) = \mathbf{Z}/2$, has generator $\langle 5 \rangle$, [16] and thus there is a unique homotopy type which contains a finite complex in dimensions $8k+3$. In general though, if $[X^{n-1}_{Q(2^n 1,1)}]$ contains a manifold then so does $\pm a^2 \cdot [X^{n-1}_{Q(2^n 1,1)}]$, and so the set of manifolds is precisely the set of finite homotopy types, so once more we get reduction.

Now

$$\sum^{L} (X^{n-1} - \text{pt}) = \vee \bigvee_p$$

where each $\bigvee_p$ is a wedge summand of $\sum^L (X_{\pi_p} - \mathrm{pt})$. See, for example, [64] or [40].

Consider the map

$$\sum^L (X^{n-1}_{\pi_p} - \mathrm{pt}) \to \sum^L X^{n-1}$$
$$\xrightarrow{\mathrm{Adj}(f)^{-L}} \Omega^{-L} B_{SG} \xrightarrow{\Omega^{-L}\phi} \Omega^{-L} B_{G/O}$$

where Ω^{-L} is an L-fold delooping. By our remarks on each $\sum^L (X^{n-1}_{\pi_p} - \mathrm{pt})$ the composite is homotopically trivial and the trivialization extends to $\sum^L X^{n-1}_{\pi_p}$. But this determines a trivialization on $\sum^L (X^{n-1} - \mathrm{pt})$ by the above remark, which gives as obstruction to trivialization a map $k: \mathbf{S}^{n-1+L} \to \Omega^{-L} B_{G/O}$. But from the construction of the trivialization on the $L+n-2$ skeleton, $(|\pi|/|\pi_p|) \cdot k \simeq 0$ for each p dividing $|\pi|$. Hence k is zero and $\mathrm{Adj}\,(\phi \cdot f) \simeq 0$ so $\phi \cdot f \simeq 0$ and f lifts to B_{SO}. □

A nice proof of 5.2 based on a transfer argument is given in [27].

COROLLARY 5.3 *Given π, X^{n-1}_π as in 5.2 there is a compact differentiable manifold M^{n-1} and a degree 1 normal map*

$$\lambda: (M^{n-1}, \nu(M)) \to (X_\pi, \xi_{X_\pi})$$

where ξ_{X_π} is some reduction of the Spivak normal bundle to X_π.

COROLLARY 5.4 *Assume that for every 2-hyperelementary subgroup $H \subset \pi$ (as above) X^{n-1}_H has the homotopy type of a differentiable manifold; then*

(1) *the surgery obstruction $s(\phi)$ vanishes for some lifting ϕ of the Spivak map;*

(2) *this problem can be chosen so that the universal cover of the resulting manifold is the ordinary sphere $\mathbf{S}^{n-1}$ for $n \geq 6$.*

(This result is due to Madsen, C. Thomas, and C. T. C. Wall [27]. The proof we give here though is simpler.)

Proof From 4.19 $s(\phi)=0$ if and only if $s(r\phi)=0$ (where r is the restriction from H to π) for every 2-hyperelementary $H \subset \pi$. From 4.12 and our assumption $s(r\phi)=0$ if and only if $s(r_0 r\phi)=0$ where $r_0: H_2 \to H$ is restriction to the 2-Sylow subgroup. But from this we have $s(\phi)=0$ if and only if $s(r_2\phi)=0$ where $r_2: \pi_2 \to \pi$ is restriction to the 2-Sylow subgroup.

For $n \equiv 2 \pmod 4$ we have that $\pi_2 = \mathbf{Z}/2^e$, but from [18] $L_1^h(\mathbf{Z}(\mathbf{Z}/2^e))=0$ so this case is correct.

Hence we may assume $n \equiv 0 \pmod 4$. But then for $\pi_2 = \mathbf{Z}(\mathbf{Z}/2^e)$ we have $L_3^h(\mathbf{Z}\pi_2)=\mathbf{Z}/2$ while $L_3^h(\mathbf{Z}Q(2^n)) = (\mathbf{Z}/2)^{n-1}$ [18]. Moreover, from 4.18 only the elements which map nontrivially under the projections $Q(2^n) \to \mathbf{Z}/2$ can be obstructions for closed manifold problems, while from 4.16 the $\mathbf{Z}/2 = L_3^h(\mathbf{Z}(\mathbf{Z}/2^e))$ maps nontrivially under the projection $\mathbf{Z}/2^e \to \mathbf{Z}/2$.

But from 4.15 all the surgery obstructions for these projections are determined using the cohomology class k in $H^*(G/O, \mathbf{Z}/2)$ via the map $g: X \to G/O$. In particular if $g^*(k_*) \equiv 0$ then the surgery obstruction is zero.

At this point we note that since k_* is primitive $(2g)^*(k_*) \equiv 0$ in mod 2 cohomology, so the surgery problem corresponding to $(2g)$ must have trivial surgery obstruction. Moreover after doing surgery the total space of the universal cover will be some homotopy sphere, and for these addition in the group of homotopy spheres Γ_{n-1} and addition of maps give the same result. Hence, since Γ_{n-1} is finite, there is some finite multiple $2\mu \cdot g$ which gives the conclusion of the corollary. □

Remark 5.5 The assumption of 5.4 is necessary and sufficient to find an action of π on $\mathbf{S}^{n-1}$. Thus the problem is completely reduced to the analysis of the surgery problem for 2-hyperelementary P-groups. But these are of two kinds $\mathbf{Z}/n \times_T \mathbf{Z}/2^e$ (type I) and $\mathbf{Z}/n \times_T Q(2^m 1, 1, 1)$ (type II).

The complete answer is known for 2-hyperelementary groups of type I. Let $\pi = A(n, 2^e, \phi)$ be a P-group of period 2^s. If $s < e$ then π acts freely and linearly on $\mathbf{S}^{2^s-1}$. If $s = e$, then π has a dihedral subgroup D_{2p} where p is an odd prime.

THEOREM 5.6 [28] *D_{2p} cannot act freely on any closed manifold M which is a $\mathbf{Z}/2$-homology sphere.*

This was reproved by R. Lee [23]. However, it is an easy consequence of 4.27. (See [13]).

The type II 2-hyperelementary groups act freely on $\mathbf{S}^{8k+7}$. hence

COROLLARY 5.7 [27] *A P-group G acts freely on some sphere if and only if G contains no dihedral subgroups.*

To elucidate Milnor's theorem we prove:

LEMMA 5.8 *Suppose G is a group whose Sylow 2-subgroup is cyclic or generalized quaternionic. Then the following are equivalent:*

(a) *Every element of order 2 is central,*
(b) *G has a unique element of order 2,*
(c) *G contains no dihedral subgroups.*

Proof (a) implies, (b) since G does not contain $\mathbf{Z}/2 \oplus \mathbf{Z}/2$; that (b) implies (c) is clear. Suppose there is a non-central element x of order 2. Then x has a conjugate y of order 2, $x \neq y$. Then $\langle x, xy \rangle$ is dihedral since $x(xy)x^{-1} = (xy)^{-1}$. □

For groups of type IIL we have the following result of Lee (which is also a consequence of 4.27):

THEOREM 5.9 *Let $\pi = Q(2^m a, b, c)$ with $m > 3$ and $b > 1$. Then π cannot act freely on any manifold M of dimension $8k+3$ which is a $\mathbf{Z}/2$-homology sphere.*

In the case $m=3$, there is no semicharacteristic obstruction to an action of $Q(8a, b, c)$ on a $\mathbf{Z}/2$-homology sphere of dimension $8k+3$, $k>0$. In fact such actions exist, [37].

Remark 5.10 In view of the results in Section 3, for every case but type II or VIb it is possible to choose X so the assumption of 5.4 is satisfied. For type VIb, however, one of the 2-hyperelementary groups is always of type II. Hence we obtain:

COROLLARY 5.11 [27]

(a) *Groups of type* I, III, IV(g), V, VI(g), *and* VI(b) *with* $p \equiv 1 \pmod 4$ *which satisfy Milnor's condition* (5.6) *act freely on a sphere in their period dimension minus* 1.

(b) *Groups of type* IV(b), VI(b) *with* $p \equiv 3 \pmod 4$ *and groups of type* II *which contain subgroups* $Q(2^n, \omega, \omega')$ *with* $n>3$ *and* ω *or* $\omega'>1$ *act freely on a sphere in twice their period dimension minus* 1 *and this is best possible.*

5.11 gives a complete answer, except for the class of 2-hyperelementary groups $\mathbf{Z}/a \times Q(8b, c, d)$. In the next section we will discuss what is known in these cases.

SECTION 6

The Surgery Problem for the Groups $Q(8a, b, c)$

Suppose $\sigma_4(Q(8a, b, c)) = 0$, and X^{8k+3} is a finite complex with $Q(8a, b, c)$ as fundamental group and universal cover $\tilde{X} \simeq \mathbf{S}^{8k+3}$. Suppose as well that over each proper subgroup π_i, $\tilde{X}^{8k+3}_{\pi_i}$ has the homotopy type of a *PL* or differentiable manifold.

The degree of the covering map

6.1 $$p_{\pi_i} : \tilde{X}^{8k+3}_{\pi_i} \to X^{8k+3}$$

is the index $[Q(8a, b, c) : \pi_i]$. Then it is easy to select a set of π_1, π_2, so the two indexes are relatively prime. Say

6.2 $$\alpha_1[\pi : \pi_1] + \alpha_2[\pi : \pi_2] = 1$$

hence

6.3 $$\coprod_1^{|\alpha_1|} M_{\pi_1} \coprod \coprod_1^{|\alpha_2|} M_{\pi_2} \to X$$

with the appropriate orientations chosen for M_{π_1}, M_{π_2} is a degree 1 map.

If it is also true that, for $\pi_3 = \pi_1 \cap \pi_2$, if the two composites

6.4 $$\begin{array}{ccccc} M_{\pi_3} & \to & M_{\pi_2} & \to & X \\ & \searrow & & \nearrow & \\ & & M_{\pi_1} & & \end{array}$$

are homotopy equivalent, then it is possible to choose a bundle ξ on X so the map in 6.3 is a degree 1 normal map.

Remark 6.5 Obvious variants of this construction can be given with any subset $\pi_1, \ldots, \pi_m$ of groups having relatively prime index in π.

For a surgery problem one actually does equivariant surgery on the $\pi = \pi_1(X)$-cover of M. But this cover is just disjoint copies of $\mathbf{S}^{8k+3}$. Specifically, the π-cover of M_{π_i} is $\coprod \mathbf{S}^{8k+3}$ consisting of $[\pi : \pi_i]$ disjoint copies. Hence for 6.3 the surgery kernel is concentrated in dimensions 0 and $8k+3$ where it is given as the kernel in the exact sequence

$$6.6 \qquad 0 \to K \to (\mathbf{Z}^{[\pi:\pi_1]})^{\alpha_1} \oplus (\mathbf{Z}^{[\pi:\pi_2]})^{\alpha_2} \xrightarrow{\varepsilon} \mathbf{Z} \to 0$$

where ε is the augmentation and the $\mathbf{Z}\pi$-module $\mathbf{Z}^{[\pi:\pi_1]}$ is induced up from the trivial π_1-module $\mathbf{Z}$. Thus the relative complex

$$6.7 \qquad C_*(\tilde{X}, \tilde{M})$$

looks like a periodic piece of a free resolution of K.

The things which are readily known about this complex are (1) its Reidemeister–DeRham torsion and (2) the quadratic forms on the middle dimensional cohomology groups of the various intermediate covers.

It turns out that in this case these are sufficient to determine the surgery obstruction. This process is worked out in some detail in [34] for the case $Q(8p, q, 1)$ with p, q distinct odd primes and $p \equiv \pm 1 \pmod 8$. However only mild changes are required to give results for the general case.

The first step is to measure the obstruction to making the complex *symmetric and even on the nose.* That is to say that we must find the obstruction to constructing an exact sequence

$$6.8 \quad 0 \to K \to C_3 \xrightarrow{d_3} C_2 \xrightarrow{\phi_2 + \phi_2^*} C_2^* \xrightarrow{d_3^*} C_3^* \to K^* \to 0$$

Then, providing the obstruiction vanishes, we try to construct such a resolution with prescribed torsion. Finally, we must see if it is possible to also match up the quadratic forms on the torsion cohomology of the various quotients of 6.8. Roughly speaking the first step gives an obstruction in $L_3^p(\mathbf{Z}\pi)$ which can be interpreted as the obstruction to finding a free π-action on $R^{8k+4}-(\mathrm{pt})$, and the second and third steps together give the exact lifting of this obstruction to $L_3^h(\mathbf{Z}\pi)$ using the Ranicki–Rothenberg sequence.

6.9
$$\cdots \to H^1(\mathbf{Z}/2, \tilde{K}_0(\mathbf{Z}\pi)) \to L_3^h(\mathbf{Z}\pi) \to L_3^p(\mathbf{Z}\pi) \to H^0(\mathbf{Z}/2, \tilde{K}_0(\mathbf{Z}\pi))$$

The details are quite technical, but a fairly clear sequence of results emerges.

THEOREM 6.10 *Let $\pi = Q(8p, q, 1)$ with p, q distinct primes and suppose $p \equiv -1 \pmod 8$; then*

(a) *π acts freely on $\mathbf{R}^{8k+4}-(\mathrm{pt})(k \geq 1)$ if and only if $q \equiv 1 \pmod 4$ and p has has odd order* (mod q).

(b) *π acts freely on $\mathbf{R}^{8k+4}-(\mathrm{pt})$ but not on $\mathbf{S}^{8k+3}(k \geq 1)$ if $q \equiv 5 \pmod 8$ and p has odd order* (mod q).

(c) *π acts freely on $\mathbf{S}^{8k+3}(k \geq 1)$ if $q \equiv 1 \pmod 8$ and p has odd order* (mod q).

Remark 6.11 In 6.10(c) the surgery obstruction also vanishes in dimension 3. But in dimension 3 surgery only gives homology equivalences. Consequently each of the groups π satisfying the conditions in 6.10(c) acts freely on a homology 3-sphere, while the groups of 6.10(a), and 6.10(b) can't even do this.

These results depend on various properties of cyclotomic units, perhaps slightly more delicate than those involved in the analysis of $\sigma_4(\pi)$, but certainly similar to them, and they show that there can be no general all or nothing types of theorems for this last class of P-groups. On the other hand, given any specific group π in this class, the results

and techniques of [31, 34] allow one (probably with the aid of a computer) to study the units in the center of $\mathbf{Z}\pi$, to determine $\sigma_4(\pi)$, $\chi(\pi)$, and the surgery obstructions $s(X^3_\pi)$ in a straightforward (almost algorithmic) manner.

Remark 6.12 Ib Madsen has also worked on the surgery problem for the groups $Q(8p, q, 1)$ [24, 25], using the intermediate Wall groups [56]. There seem to be some minor difficulties with the number theory there, although the approach is certainly feasible.

Remark 6.13 As a matter of history, Madsen gave the first example of a group $Q(24, 13, 1)$ for which $\sigma_4(\pi) = 0$ but where the surgery obstructions are all non-zero in dimensions 3 (mod 8). Then Milgram gave the first example of a group $Q(56, 113, 1)$ for which both $\sigma_4(\pi)$ and a surgery obstruction vanished in [32]. (The main *general* result in [32] had an error, but the calculations giving $Q(56, 113, 1)$ were correct.) Then Madsen showed how his techniques also gave positive results in a number of cases for p, q small.

These questions are particularly interesting in dimension 3 of course, since any closed compact 3-manifold with finite fundamental group must have a homotopy sphere as its universal cover, and hence π must be a P-group of period 2 or 4.

The complete list of period 4 groups is:

$SL_2(\mathbf{F}_3)$	(binary tetrahedral group)
$SL_2(\mathbf{F}_5)$	(binary icosahedral group)
$TL_2(\mathbf{F}_3)$	(binary octahedral group)
$Q(4k)$	(binary dihedral group)
T^*_v	
$A(m, \mathbf{Z}/2^e, \phi)$	with m odd and $\mathrm{im}\,(\mathbf{Z}/2^e) = \mathbf{Z}/2 \subset \mathrm{Aut}\,(\mathbf{Z}/m)$
$O^*_v\ (v > 1)$	(Groups of type IIL and IIM)
$Q(2^n a, b, c)$	

and direct products of the above groups with cyclic groups of relatively prime order. In this set the first four families

give the subgroups of $\mathbf{S}^3$, hence act on $\mathbf{S}^3$. The next two families act freely and linearly on $\mathbf{S}^3$ provided $e > 1$. The dihedral group does not act freely on a homotopy $\mathbf{S}^3$ by the Milnor result 5.6. Groups of type $\mathrm{II}L$ and $O_v^*(v > 1)$ do not act freely on a homotopy $\mathbf{S}^3$ by Lee's result (see also 3.16).

Moreover, the only period 2 groups are cyclic. Thus, except for the $\mathbf{Z}/w \times Q(8a, b, c)$ (b or $c > 1$), the possible finite fundamental groups of oriented closed 3-manifolds are well understood.

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